TRIEBEL-LIZORKIN SPACES AND SHEARLETS ON THE CONE IN $\mathbb{R}^2$

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Abstract. The shearlets are a special case of the wavelets with composite dilation that, among other things, have a basis-like structure and multi resolution analysis properties. These relatively new representation systems have encountered wide range of applications, generally surpassing the performance of their ancestors due to their directional sensitivity. However, little is known about their relation with spaces other than L^2. Here, we find a characterization of a kind of anisotropic inhomogeneous Triebel-Lizorkin spaces (to be defined) with the so called "shearlets on the cone" coefficients. We first prove the boundedness of the analysis and synthesis operators with the "traditional" shearlets coefficients. Then, with the development of the smooth Parseval frames of shearlets of Guo and Labate we are able to prove a reproducing identity, which was previously possible only for the L^2 case. We also find some embeddings of the (classical) dyadic spaces into these highly anisotropic spaces, and viceversa, for certain ranges of parameters. In order to keep a concise document we develop our results in the "weightless" case ($w = 1$) and give hints on how to develop the weighted case.

1. Introduction.

The traditional (separable) multidimensional wavelets are built from tensor-like products of 1-dimensional wavelets. Hence, wavelets cannot "sense" the geometry of lower dimension discontinuities. In $\mathbb{R}^d$ the number of wavelets are $2^d - 1$ for each scale. In applications it may be desirable to be able to detect more orientations having still a basis-like representation. In recent years there have been attempts to achieve this sensitivity to more orientations. Some of them include the directional wavelets or filterbanks [2], [3], the *curvelets* [8] and the *contourlets* [10], to name just a few. Regarding the contourlets, since they are built in a discrete-time setting and from a finite set of parameters, they lack of flexibility and, for some applications, assume there exist smooth spatially compactly supported functions approximating a frequency partition as that in Subsection 2.3. On the other hand, the curvelets are built on polar coordinates so their implementation is rather difficult.

In [18], Guo, Lim, Labate, Weiss and Wilson, introduced the wavelets with composite dilation. This type of representation takes full advantage of the theory of affine systems on $\mathbb{R}^n$ and therefore provides a natural transition from the continuous representation to the discrete (basis-like) setting (as in the case of wavelets). A special case of the composite dilation wavelets is that of the shearlets system which provides Parseval frames for $L^2(\mathbb{R}^2)$ or subspaces of it (depending on the discrete

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sampling of parameters, see Subsections 2.2 and 2.3). A large amount of applications of the shearlet transform not only to the image processing can be consulted in `http://www.shearlet.org`.

We characterize a new kind of *highly anisotropic inhomogeneous* Triebel-Lizorkin spaces using the "shearlets on the cone" coefficients. The line of argumentation follows the φ-transform in [13].

Following the classical definition of Triebel-Lizorkin spaces by Triebel [24], [25], Frazier-Jawerth [13], Frazier-Jawerth-Weiss [14] and their weighted counterparts in the work of Bui [6], [7]; more recently, Bownik and Ho in [5] define the *weighted anisotropic inhomogeneous Triebel-Lizorkin spaces* $\mathbf{F}_p^{\alpha,q}(A,w)$ as the collection of all $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathbf{F}_p^{\alpha,q}(A,w)} = \|f*\Phi\|_{L^p(w)} + \left\| \left(\sum_{j=1}^{\infty} (|\mathrm{det} A|^{j\alpha} \, |f*\varphi_j|)^q \right)^{1/q} \right\|_{L^p(w)} < \infty,$$

where $\Phi, \varphi \in \mathcal{S}$ with the properties that $\mathrm{supp}(\hat{\Phi})$ is compact and $\mathrm{supp}(\hat{\varphi})$ is compact and bounded away from 0 and where $A \in GL_d(\mathbb{R})$ with all of its eigenvalues > 1. Then, they show that there exists another pair $\Psi, \psi \in \mathcal{S}$ with the same properties such that

$$\overline{\hat{\Psi}(\xi)}\hat{\Phi}(\xi) + \sum_{j=1}^{\infty} \overline{\hat{\varphi}(\xi A^{-j})}\hat{\psi}(\xi A^{-j}) = 1, \quad \text{for all } \xi \in \hat{\mathbb{R}}^d,$$

which yields the representation formula

$$f = \sum_{|Q|=1} \langle f, \Phi_Q \rangle \Psi_Q + \sum_{|Q|<1} \langle f, \varphi_Q \rangle \psi_Q,$$

for any $f \in \mathcal{S}'$ with convergence in $\mathcal{S}'$ and where Q runs through the "cubes" $Q_{j,k} = A^{-j}((0,1]^d + k)$ (notice there is no shear operation as in (3.1)). This obviates (ignores) the fact that for dimension d one needs $2^d - 1$ wavelets to cover $\hat{\mathbb{R}}^d$ (or $\mathbb{R}^d$). Since the number of wavelets remains the same across scales, one can ignore the sum over the set of non-zero vertices of the cube which changes the norm only by a constant. This is not the case for the shearlets since the cardinality of the shear parameter ℓ grows with j as $\ell = -2^j, ..., 2^j$. An observation from [5] is that "in the standard dyadic case $A = 2I$, where I is the identity matrix, and then the factor $|\det A|^{j\alpha} = 2^{nj\alpha}$ in the above definition, instead of the usual $2^{j\alpha}$. Then, there is a re-scaling of the smoothness parameter α, which in the traditional case is thought of as the number of derivatives." The same happens in the setting of the "shearlets on the cone", as we will see.

Up to now, at least to our knowledge, there has been only one attempt to relate the shearlets with spaces other than L^2 as done by Dahlke, Kutyniok, Steidl and Teschke in [9]. They establish new families of smoothness spaces by means of the coorbit space theory.

The method used here can be applied to the case of spatially compact support (separable) shearlets since they are frames with irregular sampling (see [21]). This method cannot be applied to the case of the discrete shearlets (see Subsection 2.2) since the shear parameter $\ell \in \mathbb{Z}$ causes that Lemma 8.1.1 fails, avoiding a characterization of

the homogeneous case. Nevertheless, this method can be applied to higher dimensions and different anisotropic and shear matrices, as long as a kind of Lemma 8.1.1 holds.

The outline of the paper is as follows. We review the basic facts of the different shearlet transforms in Section 2 and give the pertinent results with the corresponding references. In Section 3 we set notation and give two basic lemmata regarding almost orthogonality in the "shearlets on the cone" setting. In Sections 4 and 5 we mainly follow [13], [12], [14] and [5] to 1) prove the characterization in terms of the "shearlets on the cone" coefficients and 2) prove the identity on $\mathcal{S}'$. In Section 6 we prove some embeddings of (classical) dyadic inhomogeneous Triebel-Lizorkin spaces into the highly anisotropic inhomogeneous Triebel-Lizorkin spaces, and viceversa, for a certain range of the smoothness parameter. In Section 7 we explain how to extend this work to the weighted case. Proofs for Sections 3 and 4 are given in Section 8.

2. Shearlets

The shearlets are a generalization of the wavelets which better capture the geometrical properties of functions. They are also a special case of the so-called wavelets with composite dilation (see [18]). We give a basic introduction to the construction of different type of shearlets: continuous, discrete and discrete on the cone, in the next three subsections. A point $x \in \mathbb{R}^d$ is a column vector $x = (x_1, \ldots, x_d)^t$ and a point ω in the dual $\hat{\mathbb{R}}^d$ is a row vector $\omega = (\omega_1, \ldots, \omega_d)$.

2.1. **Continuous shearlets.** A **continuous affine system** in $L^2(\mathbb{R}^d)$ is a collection of functions of the form

$$\{T_t D_M \psi : t \in \mathbb{R}^d, M \in G\},$$

where $\psi \in L^2(\mathbb{R}^d)$, T_t is the translation operator $T_t f(x) = f(x-t)$, D_M is the dilation operator $D_M f(x) = |\det M|^{-1/2} f(M^{-1}x)$ (normalized in $L^2(\mathbb{R}^d)$), and G is a subset of $GL_d(\mathbb{R})$. In the case $d = 2$, G is the 2-parameter dilation group (see [18] for an even more general definition)

$$G = \{M_{as} = \begin{pmatrix} a & \sqrt{a}s \\ 0 & \sqrt{a} \end{pmatrix} : (a, s) \in \mathbb{R}_+ \times \mathbb{R}\}.$$

The matrix M_{as} is the product $S_s A_a$ where $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ is the area preserving **shear transformation** and $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ is the **anisotropic dilation**. Assume, in addition, that ψ is given by

$$\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\xi_2}{\xi_1}), \tag{2.1}$$

for any $\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2$, $\xi_1 \neq 0$, and where ψ_1 satisfies (Calderón's admissibility condition)

$$\int_0^\infty \left|\hat{\psi}_1(a\omega)\right|^2 \frac{da}{a} = 1, \quad \text{for a.e. } \omega \in \mathbb{R},$$

and $\|\psi_2\|_{L^2(\mathbb{R})} = 1$. Then, the affine system

$$\{\psi_{ast}(x) = a^{-3/4}\psi(M_{as}^{-1}(x-t)) : a \in \mathbb{R}_+, s \in \mathbb{R}, t \in \mathbb{R}^2\},$$

is a reproducing system for $L^2(\mathbb{R}^2)$, that is,

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^\infty |\langle f, \psi_{ast}\rangle|^2 \frac{da}{a^3} ds dt,$$

for all $f \in L^2(\mathbb{R}^2)$ (see [26]).

2.2. **Discrete shearlets.** Since $L^2(\mathbb{R}^2)$ is a separable Hilbert space, it happens that, by an appropriate "sampling" of the parameters of the continuous shearlets, there exists a construction of a basis-like system for $L^2(\mathbb{R}^2)$ with a "discrete" shearlet system.

A countable family $\{e_j : j \in \mathcal{J}\}$ of elements in a separable Hilbert space $\mathbb{H}$ is called a **frame** if there exist constants $0 < A \le B < \infty$, such that $A\|f\|^2_{\mathbb{H}} \le \sum_{j\in\mathcal{J}} |\langle f, e_j\rangle|^2 \le B\|f\|^2_{\mathbb{H}}$, for all $f \in \mathbb{H}$. A frame is called **tight** if $A = B$, and is called a **Parseval frame** if $A = B = 1$. Thus, if $\{e_j : j \in \mathcal{J}\}$ is a Parseval frame for $\mathbb{H}$, then $\|f\|^2_{\mathbb{H}} = \sum_{j\in\mathcal{J}} |\langle f, e_j\rangle|^2$, for all $f \in \mathbb{H}$, which is equivalent to the reproducing formula $f = \sum_{j\in\mathcal{J}}\langle f, e_j\rangle e_j$, with convergence in $\mathbb{H}$.

With a special sampling of the parameters one can construct a Parseval frame of discrete shearlets for $L^2(\mathbb{R}^2)$ (see [18]).

2.3. **Discrete shearlets on the cone.** In spite of the Parseval frame property for $L^2(\mathbb{R}^2)$ of the discrete shearlets system, there are no "equivalent information" among the "mostly horizontal" and "mostly vertical" shearlets (important in applications) since the tiling of $\hat{\mathbb{R}}^2$ is not "homogeneous" in these directions. The covering of $\hat{\mathbb{R}}^2$ by the discrete shearlets is done firstly by vertical bands or strips related to mostly horizontal dilations (indexed by j). Then, each band is covered by infinitely countable shear (area preserving) transformations (indexed by ℓ). With a little modification on the discrete shearlet system above one can obtain a Parseval frame for functions in $L^2(\mathbb{R}^2)$ whose Fourier transform is supported in the horizontal cone

$$\mathcal{D}^h = \{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : |\xi_1| \ge \frac{1}{8}, \left|\frac{\xi_2}{\xi_1}\right| \le 1\}. \tag{2.2}$$

Let now $\hat{\psi}_1, \hat{\psi}_2 \in C^\infty(\mathbb{R})$ with supp $\hat{\psi}_1 \subset [-\frac{1}{2}, -\frac{1}{16}] \cup [\frac{1}{16}, \frac{1}{2}]$ and supp $\hat{\psi}_2 \subset [-1, 1]$ such that

$$\sum_{j\ge 0} \left|\hat{\psi}_1(2^{-2j}\omega)\right|^2 = 1, \quad \text{for } |\omega| \ge \frac{1}{8} \tag{2.3}$$

and

$$\left|\hat{\psi}_2(\omega - 1)\right|^2 + \left|\hat{\psi}_2(\omega)\right|^2 + \left|\hat{\psi}_2(\omega + 1)\right|^2 = 1, \quad \text{for } |\omega| \le 1. \tag{2.4}$$

It follows from (2.4) that, for $j \ge 0$,

$$\sum_{\ell=-2^j}^{2^j} \left|\hat{\psi}_2(2^j\omega - \ell)\right|^2 = 1, \quad \text{for } |\omega| \le 1. \tag{2.5}$$

Let

$$A_h = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $\hat{\psi}^h(\xi) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\frac{\xi_2}{\xi_1})$. From (2.3) and (2.5) it follows that

$$\begin{aligned}\sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}\left|\hat{\psi}^h(\xi A_h^{-j}B_h^{-\ell})\right|^2 &= \sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}\left|\hat{\psi}_1(2^{-2j}\xi_1)\right|^2\left|\hat{\psi}_2(2^j\frac{\xi_2}{\xi_1}-\ell)\right|^2 \\ &= \sum_{j\geq 0}\left|\hat{\psi}_1(2^{-2j}\xi_1)\right|^2\sum_{\ell=-2^j}^{2^j}\left|\hat{\psi}_2(2^j\frac{\xi_2}{\xi_1}-\ell)\right|^2 = 1, \quad (2.6)\end{aligned}$$

for $\xi = (\xi_1, \xi_2) \in \mathcal{D}^h$ and which we will call the **Parseval frame condition** (for the horizontal cone). Since supp $\hat{\psi}^h \subset [-\frac{1}{2}, \frac{1}{2}]^2$, (2.6) implies that the shearlet system

$$\{\psi^h_{j,\ell,k}(x) = 2^{3j/2}\psi^h(B_h^\ell A_h^j x - k) : j \geq 0, -2^j \leq \ell \leq 2^j, k \in \mathbb{Z}^2\}, \qquad (2.7)$$

is a Parseval frame for $L^2((\mathcal{D}^h)^\vee) = \{f \in L^2(\mathbb{R}^2) : \text{supp } \hat{f} \subset \mathcal{D}^h\}$ (see [18], Subsection 5.2.1). This means that

$$\sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}\sum_{k\in\mathbb{Z}^2}\left|\langle f, \psi^h_{j,\ell,k}\rangle\right|^2 = \|f\|^2_{L^2(\mathbb{R}^2)},$$

for all $f \in L^2(\mathbb{R}^2)$ such that supp $\hat{f} \subset \mathcal{D}^h$. There are several examples of functions ψ_1, ψ_2 satisfying the properties described above (see [16]). Since $\hat{\psi}^h \in C_c^\infty(\hat{\mathbb{R}}^2)$, there exist C_N such that $\left|\psi^h(x)\right| \leq C_N(1+|x|)^{-N}$ for all $N \in \mathbb{N}$. The geometric properties of the horizontal shearlets system are more evident by observing that

$$\text{supp } (\psi_{j,\ell,k})^\wedge \subset \{\xi \in \hat{\mathbb{R}}^2 : \xi_1 \in [-2^{2j-1}, -2^{2j-4}] \cup [2^{2j-4}, 2^{2j-1}], \left|\frac{\xi_2}{\xi_1} - \ell 2^{-j}\right| \leq 2^{-j}\}.$$

One can also construct a Parseval frame for the vertical cone

$$\mathcal{D}^v = \{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : |\xi_2| \geq \frac{1}{8}, \left|\frac{\xi_1}{\xi_2}\right| \leq 1\},$$

by defining $\hat{\psi}^v(\xi) = \hat{\psi}_1(\xi_2)\hat{\psi}_2(\frac{\xi_1}{\xi_2})$ and with anisotropic and shear matrices

$$A_v = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad B_v = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let $\hat{\varphi} \in C_c^\infty(\mathbb{R}^2)$, with supp $\hat{\varphi} \subset [-\frac{1}{4}, \frac{1}{4}]^2$ and $|\hat{\varphi}| = 1$ for $\xi \in [-\frac{1}{8}, \frac{1}{8}]^2 = \mathcal{R}$, be such that

$$\begin{aligned}P(\xi) &= |\hat{\varphi}(\xi)|^2\chi_{\mathcal{R}}(\xi) + \sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}\left|\hat{\psi}^h(\xi A_h^{-j}B_h^{-\ell})\right|^2\chi_{\mathcal{D}^h}(\xi) \\ &\quad + \sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}\left|\hat{\psi}^v(\xi A_v^{-j}B_v^{-\ell})\right|^2\chi_{\mathcal{D}^v}(\xi) = 1, \quad \text{for all } \xi \in \hat{\mathbb{R}}^2. \qquad (2.8)\end{aligned}$$

3. Notation and almost orthogonality

Since all results in the horizontal cone $\mathcal{D}^h$ can be stated for the vertical one $\mathcal{D}^v$, with the obvious modifications as explained in Subsection 2.3, we drop the superindex h and develop only for the horizontal cone and refer only to "the cone".

We will develop our results with

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

an anisotropic dilation and a shear matrix, respectively. We consider ψ defined by $\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi_2/\xi_1)$ with ψ_1 and ψ_2 satisfying (2.3), (2.4) and (2.5). In order to follow [13] we will require $\varphi, \psi \in \mathcal{S}$ with the same conditions on supp $(\hat{\varphi})$, supp $(\hat{\psi}_1)$, supp $(\hat{\psi}_2)$ and equations (2.3), (2.4) and (2.5) in order to preserve the geometrical properties of anisotropic and shear operations. Following the notation for the usual isotropic dilation, $\varphi_t(x) := \frac{1}{t}\varphi(\frac{x}{t})$, we denote for a matrix $M \in GL_2(\mathbb{R})$ the anisotropic dilation $\varphi_M(x) = |\det M|^{-1} \varphi(M^{-1}x)$ (do not confuse with the dilation operator normalized in L^2 as in Subsection 2.1). We also denote $\tilde{\varphi}(x) = \overline{\varphi(-x)}$. For $Q_0 = [0,1)^2$, write

$$Q_{j,\ell,k} = A^{-j}B^{-\ell}(Q_0 + k), \tag{3.1}$$

with $j \geq 0$, $\ell = -2^j, \ldots, 2^j$ and $k \in \mathbb{Z}^2$. Therefore, $\int \chi_{Q_{j,\ell,k}} = |Q_{j,\ell,k}| = |Q_{j,\ell}| = |Q_j| = 2^{-3j} = |\det A|^{-j}$. We also write $\tilde{\chi}_Q(x) = |Q|^{-1/2} \chi_Q(x)$. Let $\mathcal{Q}_{AB} := \{Q_{j,\ell,k} : j \geq 0, \ell = -2^j, \ldots, 2^j, k \in \mathbb{Z}^2\}$ and $\mathcal{Q}^{j,\ell} := \{Q_{j,\ell,k} : k \in \mathbb{Z}^2\}$, then $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^2$. To shorten notation and clear exposition, we will identify the multi indices (j,ℓ,k) and (i,m,n) with P and Q, respectively. This way we write $\psi_P = \psi_{j,\ell,k}$ or $\psi_Q = \psi_{i,m,n}$. Also, we let x_P and x_Q be the lower left corners $A^{-j}B^{-\ell}k$ and $A^{-i}B^{-m}n$ of the "cubes" $P = Q_{j,\ell,k}$ and $Q = Q_{i,m,n}$, respectively. Let $B_r(x)$ be the Euclidean ball centered in x with radius r.

The elements of the affine collection

$$\mathcal{A}_{AB} := \{\psi_{j,\ell,k}(x) = |\det A|^{j/2} \psi(B^\ell A^j x - k) : j \geq 0, -2^j \leq \ell \leq 2^j, k \in \mathbb{Z}^2\},$$

have Fourier transform

$$(\psi_{j,\ell,k})^\wedge(\xi) = |\det A|^{-j/2} \hat{\psi}(\xi A^{-j}B^{-\ell})\mathbf{e}^{-2\pi i \xi A^{-j}B^{-\ell}k}.$$

Using the anisotropic dilation it is also easy to verify that

$$\psi_{A^{-j}B^{-\ell}}(x - A^{-j}B^{-\ell}k) = |\det A|^{j/2} \psi_{j,\ell,k}(x) = |P|^{-1/2} \psi_P(x)$$

and thus

$$\left(\psi_{A^{-j}B^{-\ell}}(\cdot - A^{-j}B^{-\ell}k)\right)^\wedge(\xi) = \hat{\psi}(\xi A^{-j}B^{-\ell})\mathbf{e}^{-2\pi i \xi A^{-j}B^{-\ell}k}.$$

We also have

$$\begin{aligned} \langle f, \psi_P \rangle &= \langle f, \psi_{j,\ell,k} \rangle \\ &= \int_{\mathbb{R}^2} f(x)\overline{2^{-3j/2}\psi_{A^{-j}B^{-\ell}}(x - A^{-j}B^{-\ell}k)}dx \\ &= |P|^{1/2} (f * \tilde{\psi}_{A^{-j}B^{-\ell}})(x_P). \end{aligned} \tag{3.2}$$

3.1. **Almost Orthogonality.** From the support condition on $\hat{\psi}_1$, the definition of the matrix A and (2.3), the set of all shearlets at scale j (for all shear and translation parameters) interacts with the sets of all shearlets only at scales $j-1, j$ and $j+1$ (for all shear and translation parameters). The next result (for functions in $\mathcal{S}$ not necessarily shearlets) is proved in Subsection 8.1.

Lemma 3.1.1. *Let $g, h \in \mathcal{S}$. For $i = j-1,\ j,\ j+1 \geq 0$, let Q be identified with (i, m, n). Then, for every $N > 2$, there exists a $C_N > 0$ such that*

$$|g_{A^{-j}B^{-\ell}} * h_Q(x)| \leq \frac{C_N\, |Q|^{-\frac{1}{2}}}{(1 + 2^i\, |x - x_Q|)^N},$$

for all $x \in \mathbb{R}^2$.

By construction, for the specific case of the "shearlets on the cone" we even have the next more informative property stated in the Fourier domain. The next result is also proved in Subsection 8.1.

Lemma 3.1.2. *Let supp $\hat{\psi}$ be as in Subsection 2.3. Then, the support of a horizontal $(\psi_{j,\ell,k})^\wedge$ overlaps with the support of at most 11 other horizontal $(\psi_{i,m,n})^\wedge$ for $(j, \ell) \neq (i, m)$ and all $k, n \in \mathbb{Z}^2$.*

Remark 3.1.3. *Since the translation parameters k and n do not affect the support in the frequency domain, then for*

$$f = T_\psi \mathbf{s} = \sum_{Q \in \mathcal{Q}_{AB}} s_Q \psi_Q = \sum_{i \geq 0} \sum_{m=-2^i, \ldots, 2^i} \sum_{n \in \mathbb{Z}^2} s_{i,m,n} \psi_{i,m,n},$$

we formally have that

$$(\tilde{\psi}_{A^{-j}B^{-\ell}} * f)(x) = \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \sum_{Q \in \mathcal{Q}^{i,m}} s_Q (\tilde{\psi}_{A^{-j}B^{-\ell}} * \psi_Q)(x),$$

where $m(\ell, i)$ are the shear indices of those shearlets in the Fourier domain "surrounding" the support of $(\tilde{\psi}_{A^{-j}B^{-\ell}})^\wedge$ and the sum $\sum_{i=j-1}^{j+1} \sum_{m(\ell,i)}$ has at most 11+1 terms for all j by Lemma 3.1.2.

Remark 3.1.4. *From Lemma 3.1.2 the number of horizontal/vertical shearlets overlapping on the Fourier domain is bounded for all scales, since the vertical system for $\mathcal{D}^v$ is an orthonormal rotation of the horizontal system for $\mathcal{D}^h$, leaving all distances and angles of the supports unaltered.*

4. The characterization

After defining the distribution spaces we will work on, we will ignore the "directions" of the horizontal and vertical ($\mathfrak{d} = \{h, v\}$) shearlets as done in the wavelets case. We will also ignore the coarse function φ and associated sequence since they are already treated in the literature (see Section 12 in [13]).

4.1. AB**-anisotropic inhomogeneous Triebel-Lizorkin spaces.** Let $\varphi, \psi \in \mathcal{S}$ be as in Subsection 2.3.

Definition 4.1.1. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. The AB-**anisotropic inhomogeneous** Triebel-Lizorkin **distribution** spaces $\mathbf{F}_p^{\alpha,q}(AB)$ are defined as the collection of all $f \in \mathcal{S}'$ such that*

$$\begin{aligned} \|f\|_{\mathbf{F}_p^{\alpha,q}(AB)} &= \|f * \varphi\|_{L^p} \\ &+ \left\| \left(\sum_{\mathfrak{d}=\{h,v\}} \left\{ \sum_{j\geq 0} \sum_{\ell=-2^j}^{2^j} [|Q_j|^{-\alpha} \left| \tilde{\psi}^{\mathfrak{d}}_{A_{\mathfrak{d}}^{-j} B_{\mathfrak{d}}^{-\ell}} * f \right|]^q \right\} \right)^{1/q} \right\|_{L^p} < \infty. \end{aligned} \quad (4.1)$$

To work in the sequence level with the shearlets coefficients we also have the next definition.

Definition 4.1.2. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. The AB-**anisotropic inhomogeneous** Triebel-Lizorkin **sequence** spaces $\mathbf{f}_p^{\alpha,q}(AB)$ are defined as the collection of all complex-valued sequences $\mathbf{s} = \{s_Q\}_{Q \in \mathcal{Q}_{AB}}$ such that*

$$\|\mathbf{s}\|_{\mathbf{f}_p^{\alpha,q}(AB)} = \left\| \left(\sum_{Q \in \mathcal{Q}_{AB}} (|Q|^{-\alpha} |s_Q| \, \tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^p} < \infty. \quad (4.2)$$

We also formally define the **analysis** and **synthesis operators** as

$$S_\psi f = \{\langle f, \psi_Q \rangle\}_{Q \in \mathcal{Q}_{AB}} \quad \text{and} \quad T_\psi \mathbf{s} = \sum_{Q \in \mathcal{Q}_{AB}} s_Q \psi_Q, \quad (4.3)$$

respectively.

Remark 4.1.3. *Observe that in (4.1) there are no trace of the characteristic functions $\chi_{\mathcal{D}^h}$, $\chi_{\mathcal{D}^v}$ and $\chi_{\mathcal{R}}$ (in the Fourier domain) which enable the identity in $L^2(\mathbb{R}^2)$ via (2.8). Instead, we will simply bound the operators, since ignoring $\chi_{\mathcal{D}^h}$, $\chi_{\mathcal{D}^v}$ and $\chi_{\mathcal{R}}$ in (2.8) affects only the Parseval condition on the frame (see Lemma 3.1.2 and Remark 3.1.4).*

4.2. **Two basic results.** As aforementioned, for the proof of our main result (Theorem 4.3.1) we follow [13]. This is based on a kind of Peetre's inequality to bound $S_\psi : \mathbf{F}_p^{\alpha,q}(AB) \to \mathbf{f}_p^{\alpha,q}(AB)$, and a characterization of $\mathbf{f}_p^{\alpha,q}(AB)$ to bound $T_\psi : \mathbf{f}_p^{\alpha,q}(AB) \to \mathbf{F}_p^{\alpha,q}(AB)$. We start with a definition and a well known result.

Definition 4.2.1. *The **Hardy-Littlewood maximal function**, $\mathcal{M}f(x)$, is given by*

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy,$$

for a locally integrable function f on $\mathbb{R}^2$ and where $B_r(x)$ is the ball with center in x and radius r.

It is well known that $\mathcal{M}$ is bounded on L^p, $1 < p \leq \infty$. It is also true that the next vector-valued inequality holds (see [11]).

Theorem 4.2.2. ***[Fefferman-Stein]*** *For $1 < p < \infty$ and $1 < q \leq \infty$, there exists a constant $C_{p,q}$ such that*

$$\left\| \left\{ \sum_{i=1}^{\infty} (\mathcal{M} f_i)^q \right\}^{1/q} \right\|_{L^p} \leq C_{p,q} \left\| \left\{ \sum_{i=1}^{\infty} f_i^q \right\}^{1/q} \right\|_{L^p},$$

for any sequence $\{f_i : i = 1, 2, \ldots\}$ of locally integrable functions.

Let us define the ***shear anisotropic*** **Peetre's maximal function**. For all $\lambda > 0$,

$$(\psi^{**}_{j,\ell,\lambda} f)(x) := \sup_{y\in\mathbb{R}^2} \frac{|(\psi_{A^{-j}B^{-\ell}} * f)(x-y)|}{(1+|B^\ell A^j y|)^{2\lambda}}. \tag{4.4}$$

We then have a ***shear anisotropic*** **Peetre's inequality**. Next lemma is proved in Subsection 8.2.

Lemma 4.2.3. *Let ψ be band limited and $f \in \mathcal{S}'$. Then, for any real $\lambda > 0$, there exists a constant C_λ such that*

$$(\psi^{**}_{j,\ell,\lambda} f)(x) \leq C_\lambda \left\{ \mathcal{M}(|\psi_{A^{-j}B^{-\ell}} * f|^{1/\lambda})(x) \right\}^\lambda, \quad x \in \mathbb{R}^2.$$

Identify Q and P with (i, m, n) and (j, ℓ, k), respectively. For all $r > 0$, $N \in \mathbb{N}$ and $i \geq j \geq 0$, define

$$(s^*_{r,N})_Q := \left(\sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j|x_Q - x_P|)^N} \right)^{1/r},$$

and $\mathbf{s}^*_{r,N} = \{(s^*_{r,N})_Q\}_{Q\in\mathcal{Q}_{AB}}$. We then have the characterization of the sequence spaces $\mathbf{f}^{\alpha,q}_p(AB)$ in terms of $\mathbf{s}^*_{r,N}$ which is used to prove the boundedness of $T_\psi : \mathbf{f}^{\alpha,q}_p(AB) \to \mathbf{F}^{\alpha,q}_p(AB)$. Next result is also proved in Subsection 8.2.

Lemma 4.2.4. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then, for all $r > 0$ and $N > 3\max(1, r/q, r/p)$ there exists $C > 0$ such that*

$$\|\mathbf{s}\|_{\mathbf{f}^{\alpha,q}_p(AB)} \leq \left\|\mathbf{s}^*_{r,N}\right\|_{\mathbf{f}^{\alpha,q}_p(AB)} \leq C\,\|\mathbf{s}\|_{\mathbf{f}^{\alpha,q}_p(AB)}.$$

4.3. Boundedness of S_ψ and T_ψ.

As previously mentioned, since we are leaving aside the characteristic functions $\chi_{\mathcal{D}^h}$, $\chi_{\mathcal{D}^v}$ and $\chi_{\mathcal{R}}$ of (2.8) one cannot hope for a reproducing identity for the spaces $\mathbf{F}^{\alpha,q}_p(AB)$.

Theorem 4.3.1. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then, the operators $S_\psi : \mathbf{F}^{\alpha,q}_p(AB) \to \mathbf{f}^{\alpha,q}_p(AB)$ and $T_\psi : \mathbf{f}^{\alpha,q}_p(AB) \to \mathbf{F}^{\alpha,q}_p(AB)$ are well defined and bounded.*

Proof. We prove only the case $q < \infty$. To prove the boundedness of S_ψ suppose $f \in \mathbf{F}^{\alpha,q}_p(AB)$. Let P be identified with (j,ℓ,k). Then, $\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x_P)\right| \chi_P = |\langle f, \psi_P\rangle|\,\tilde{\chi}_P$, as in (3.2). Let $E = \cup_{\kappa\in K} Q_{j,\ell,\kappa}$ where $K = \{(0,0), (-1,0), (0,-1), (-1,-1)\}$. Since $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^2$ we have for $x \in P'$ and $P' \in \mathcal{Q}^{j,\ell}$,

$$\sum_{P\in\mathcal{Q}^{j,\ell}} \left[|P|^{-\alpha}\,|(S_\psi f)_P|\,\tilde{\chi}_P(x)\right]^q$$

$$\begin{aligned}
&= |\det A|^{j\alpha q} \sum_{P\in\mathcal{Q}^{j,\ell}} \left[\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x_P)\right| \chi_P(x)\right]^q \\
&\le |\det A|^{j\alpha q} \sum_{P\in\mathcal{Q}^{j,\ell}} \sup_{y\in P} \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(y)\right|^q \chi_P(x) \\
&\le |\det A|^{j\alpha q} \sup_{z\in E} \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x-z)\right|^q \\
&= |\det A|^{j\alpha q} \sup_{z\in E} \left[\frac{\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x-z)\right|}{(1+|B^\ell A^j z|)^{2/\lambda}}\right]^q (1+\left|B^\ell A^j z\right|)^{q2/\lambda} \\
&\le |\det A|^{j\alpha q} \left[\sup_{z\in\mathbb{R}^2} \frac{\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x-z)\right|}{(1+|B^\ell A^j z|)^{2/\lambda}}\right]^q \sup_{\kappa\in K}(1+\mathrm{Diam}(Q_{0,0,\kappa}))^{2q/\lambda} \\
&= C_{q,\lambda} \, |\det A|^{j\alpha q} \, (\tilde{\psi}^{**}_{j,\ell,1/\lambda} f)^q(x) \\
&\le C_{q,\lambda} \, |\det A|^{j\alpha q} \left\{\mathcal{M}\left(\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|^\lambda\right)(x)\right\}^{q/\lambda},
\end{aligned}$$

because of Lemma 4.2.3 (with $1/\lambda$ instead of λ in the last inequality). Now, take $0 < \lambda < \min(p, q)$. Then, the previous estimate and Theorem 4.2.2 yield

$$\begin{aligned}
\|S_\psi f\|_{\mathbf{f}^{\alpha,q}_p(AB)} &= \left\| \left(\sum_{j\ge 0} \sum_{\ell=-2^j}^{2^j} \sum_{P\in\mathcal{Q}^{j,\ell}} [|P|^{-\alpha} \left|(S_\psi f)_P\right| \tilde{\chi}_P]^q \right)^{1/q} \right\|_{L^p} \\
&\le C \left\| \left(\sum_{j\ge 0} \sum_{\ell=-2^j}^{2^j} \left\{ \mathcal{M}\left(|\det A|^{j\alpha\lambda} \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|^\lambda \right) \right\}^{q/\lambda} \right)^{1/q} \right\|_{L^p} \\
&= C \left\| \left(\sum_{j\ge 0} \sum_{\ell=-2^j}^{2^j} \left\{ \mathcal{M}\left(|\det A|^{j\alpha\lambda} \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|^\lambda \right) \right\}^{q/\lambda} \right)^{\lambda/q} \right\|_{L^{p/\lambda}}^{1/\lambda} \\
&\le C \left\| \left(\sum_{j\ge 0} \sum_{\ell=-2^j}^{2^j} |\det A|^{j\alpha q} \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|^q \right)^{\lambda/q} \right\|_{L^{p/\lambda}}^{1/\lambda} \\
&= C \left\| \left(\sum_{j\ge 0} \sum_{\ell=-2^j}^{2^j} [|\det A|^{j\alpha} \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|]^q \right)^{1/q} \right\|_{L^p} \\
&= C \, \|f\|_{\mathbf{F}^{\alpha,q}_p(AB)} .
\end{aligned}$$

To prove the boundedness of T_ψ suppose $\mathbf{s} = \{s_Q\}_Q \in \mathbf{f}^{\alpha,q}_p$ and $f = T_\psi \mathbf{s} = \sum_{Q\in\mathcal{Q}_{AB}} s_Q \psi_Q$. By Lemma 3.1.2 (see also Remark 3.1.3) and Lemma 3.1.1, we have

for $x \in Q'$ and $Q' \in \mathcal{Q}^{i,m}$,

$$\begin{aligned}
\left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f(x)\right| &\leq \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \sum_{Q\in\mathcal{Q}^{i,m}} |s_Q| \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * \psi_Q(x)\right| \\
&\leq C \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \sum_{Q\in\mathcal{Q}^{i,m}} |s_Q| \frac{|Q|^{-1/2}}{(1+2^i|x-x_Q|)^N} \\
&\leq C' \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \sum_{Q\in\mathcal{Q}^{i,m}} |s_Q| \frac{|Q|^{-1/2}}{(1+2^i|x_{Q'}-x_Q|)^N} \\
&= C' \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} |Q|^{-1/2} (s^*_{1,N})_{Q'} \chi_{Q'}(x) \\
&= C' \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \sum_{Q\in\mathcal{Q}^{i,m}} (s^*_{1,N})_Q \tilde{\chi}_Q(x),
\end{aligned}$$

for all $N > 2$ and because $\mathcal{Q}^{i,m}$ is a partition of $\mathbb{R}^2$. Let $N > 3\max(1, 1/q, 1/p)$. Then, since the pair (i,m) runs over each pair of scale and shear parameters at most 12 times by Lemma 3.1.2, the previous estimate yields

$$\begin{aligned}
\|T_\psi \mathbf{s}\|_{\mathbf{F}_p^{\alpha,q}(AB)} &= \left\| \left(\sum_{j\geq 0} \sum_{\ell=-2^j}^{2^j} [|Q_j|^{-\alpha} \left|\tilde{\psi}_{A^{-j}B^{-\ell}} * f\right|]^q \right)^{1/q} \right\|_{L^p} \\
&\leq C \left\| \left(\sum_{j\geq 0} \sum_{\ell=-2^j}^{2^j} \left[|Q_j|^{-\alpha} \sum_{i=j-1}^{j+1} \sum_{m(\ell,i)} \sum_{Q\in\mathcal{Q}^{i,m}} (s^*_{1,N})_Q \tilde{\chi}_Q \right]^q \right)^{1/q} \right\|_{L^p} \\
&\leq C \left\| \left(\sum_{j\geq 0} \sum_{\ell=-2^j}^{2^j} \left[\sum_{Q\in\mathcal{Q}^{j,\ell}} |Q|^{-\alpha} (s^*_{1,N})_Q \tilde{\chi}_Q \right]^q \right)^{1/q} \right\|_{L^p} \\
&= C \left\| \left(\sum_{j\geq 0} \sum_{\ell=-2^j}^{2^j} \sum_{Q\in\mathcal{Q}^{j,\ell}} [|Q|^{-\alpha} (s^*_{1,N})_Q \tilde{\chi}_Q]^q \right)^{1/q} \right\|_{L^p} \\
&= C \left\|\mathbf{s}^*_{1,N}\right\|_{\mathbf{f}_p^{\alpha,q}(AB)} \leq C \|\mathbf{s}\|_{\mathbf{f}_p^{\alpha,q}(AB)},
\end{aligned}$$

because $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^2$ and Lemma 4.2.4 in the last inequality.

■

Remark 4.3.2. *With the same arguments as in Remark 2.6 in* [13], *the definition of* $\mathbf{F}_p^{\alpha,q}(AB)$ *is independent of the choice of* $\psi \in \mathcal{S}$ *as long as it satisfies the requirements in Subsection 2.3.*

5. The identity with smooth Parseval frames

Recently, Guo and Labate in [17] found a way to overcome the use of characteristic functions in the Fourier domain to restrict the horizontal/vertical shearlets to the respective cone (see (2.8)). The use of these characteristic functions affects the smoothness of the *boundary shearlets* (those with $\ell = \pm 2^j$). They slightly modify the definition of these boundary shearlets instead of projecting them into the cone. This new shearlets system is not affine-like. However, they do produce the same frequency tiling as that in Subsection 2.3.

5.1. **The new smooth shearlets system.** This subsection is a brief summary of some results in [17] and is intended to show the construction of such smooth Parseval frames. Let ϕ be a C^∞ univariate function such that $0 \leq \phi \leq 1$, with $\hat{\phi} = 1$ on $[-1/16, 1/16]$ and $\hat{\phi} = 0$ outside $[-1/8, 1/8]$ (*i.e.*, ϕ is a rescaled Meyer wavelet). For $\xi \in \hat{\mathbb{R}}^2$, let $\hat{\Phi}(\xi) = \hat{\phi}(\xi_1)\hat{\phi}(\xi_2)$ and $W^2(\xi) = \hat{\Phi}^2(2^{-2}\xi) - \hat{\Phi}^2(\xi)$. It follows that

$$\hat{\Phi}(\xi) + \sum_{j\geq 0} W^2(2^{-2j}\xi) = 1, \quad \text{for all } \xi \in \hat{\mathbb{R}}^2.$$

Let now $v \in C^\infty(\mathbb{R})$ be such that $v(0) = 1$, $v^{(n)}(0) = 0$ for all $n \geq 1$, supp $v \subset [-1, 1]$ and

$$|v(u-1)|^2 + |v(u)|^2 + |v(u+1)|^2 = 1, \ \ |u| \leq 1.$$

Then, for any $j \geq 0$,

$$\sum_{m=-2^j}^{2^j} \left|v(2^j u - m)\right|^2 = 1, \ \ |u| \leq 1.$$

See Subsection 2.3 for comments on the construction of theses functions and similar properties.

With $V_h(\xi_1, \xi_2) = v(\frac{\xi_2}{\xi_1})$, $\xi \in \mathcal{D}^h$, the horizontal shearlet system for $L^2(\mathbb{R}^2)$ is defined as the countable collection of functions

$$\{\psi^h_{j,\ell,k} : j \geq 0, |\ell| < 2^j, k \in \mathbb{Z}^2\},$$

whose elements are defined by their Fourier transform

$$(\psi^h_{j,\ell,k})^\wedge(\xi) = |\det A_h|^{-j/2}\, W(2^{-2j}\xi) V_h(\xi A_h^{-j} B_h^{-\ell}) \mathbf{e}^{-2\pi i \xi A_h^{-j} B_h^{-\ell} k}, \quad \xi \in \mathcal{D}^h, \tag{5.1}$$

where A_h and B_h are as in Subsection 2.3. Similarly, one can construct the vertical shearlet system as in Subsection 2.3.

For the *boundary shearlets* let $j \geq 1$, $\ell = \pm 2^j$ and $k \in \mathbb{Z}^2$, then $(\psi_{j,\ell,k})^\wedge(\xi) = 2^{-\frac{3}{2}j - \frac{1}{2}} W(2^{-2j}\xi) v(2^j \frac{\xi_2}{\xi_1} - \ell) \mathbf{e}^{-2\pi i \xi 2^{-1} A_h^{-j} B_h^{-\ell} k}$ for $\xi \in \mathcal{D}^h$, and $(\psi_{j,\ell,k})^\wedge(\xi) = 2^{-\frac{3}{2}j - \frac{1}{2}} W(2^{-2j}\xi) v(2^j \frac{\xi_1}{\xi_2} - \ell) \mathbf{e}^{-2\pi i \xi 2^{-1} A_v^{-j} B_v^{-\ell} k}$ for $\xi \in \mathcal{D}^v$. When $j = 0$, $\ell = \pm 1$ and $k \in \mathbb{Z}^2$ define $(\psi_{0,\ell,k})^\wedge(\xi) = W(\xi) v(\frac{\xi_2}{\xi_1} - \ell) \mathbf{e}^{-2\pi i \xi k}$ for $\xi \in \mathcal{D}^h$, and $(\psi_{0,\ell,k})^\wedge(\xi) = W(\xi) v(\frac{\xi_1}{\xi_2} - \ell) \mathbf{e}^{-2\pi i \xi k}$ for $\xi \in \mathcal{D}^v$. These boundary shearlets are also $C^\infty(\hat{\mathbb{R}}^2)$ (see [17]). This new system is not affine-like since the function W is not shear-invariant. However, as previously mentioned, they generate the same frequency tiling.

The new smooth Parseval frame condition is now written as (see Theorem 2.3 in [17])

$$\left|\hat{\Phi}(\xi)\right|^2 + \sum_{\mathfrak{d}=1}^{2}\sum_{j\geq 0}\sum_{|\ell|<2^j}\left|\hat{\psi}^{\mathfrak{d}}(\xi A_{\mathfrak{d}}^{-j}B_{\mathfrak{d}}^{-\ell})\right|^2 + \sum_{j\geq 0}\sum_{\ell=\pm 2^j}\left|\hat{\psi}(\xi A^{-j}B^{-\ell})\right|^2 = 1, \qquad (5.2)$$

for all $\xi \in \hat{\mathbb{R}}^2$ and where $\mathfrak{d} = 1, 2$ stands for horizontal and vertical directions and we omit the subindex for the matrices of the boundary shearlets. Notice that now there do not exist characteristic functions as in (2.8).

5.2. **The reproducing identity on $\mathcal{S}'$.** Our goal is to show that, with the smooth Parseval frames of shearlets of Guo and Labate in [17], $T_\psi \circ S_\psi$ is the identity on $\mathcal{S}'$ and, therefore, on $\mathbf{F}_p^{\alpha,q}(AB)$. First, we show that any $f \in \mathcal{S}'$ admits a kind of Littlewood-Paley decomposition with shear anisotropic dilations, for which we follow [5]. Then, we show the reproducing identity in $\mathcal{S}'$ following [14]. Denote $\hat{\Phi} = (\psi_{-1})^\wedge$.

Lemma 5.2.1. *Let $\{\psi_{j,\ell,k} : j \geq 0, \ell = -2^j, \dots, 2^j, k \in \mathbb{Z}^2\}$ be the smooth shearlet system that verifies (5.2). Then, for any $f \in \mathcal{S}'$,*

$$\begin{aligned} f = f * \tilde{\psi}_{-1} * \psi_{-1} \quad &+ \quad \sum_{\mathfrak{d}=1}^{2}\sum_{j\geq 0}\sum_{|\ell|<2^j} f * \tilde{\psi}^{\mathfrak{d}}_{A_{\mathfrak{d}}^{-j}B_{\mathfrak{d}}^{-\ell}} * \psi^{\mathfrak{d}}_{A_{\mathfrak{d}}^{-j}B_{\mathfrak{d}}^{-\ell}} \\ &+ \quad \sum_{j\geq 0}\sum_{\ell=\pm 2^j} f * \tilde{\psi}_{A^{-j}B^{-\ell}} * \psi_{A^{-j}B^{-\ell}}, \end{aligned}$$

with convergence in $\mathcal{S}'$.

Proof. One can see Peetre's discussion on pp. 52-54 of [23] regarding convergence. Since the Fourier transform $\mathcal{F}$ is an isomorphism of $\mathcal{S}'$, it suffices to show that

$$\begin{aligned} \hat{f}(\xi) = \hat{f}(\xi)\left|(\psi_{-1})^\wedge(\xi)\right|^2 \quad &+ \quad \sum_{\mathfrak{d}=1}^{2}\sum_{j\geq 0}\sum_{|\ell|<2^j} \hat{f}(\xi)\left|\hat{\psi}^{\mathfrak{d}}(\xi A_{\mathfrak{d}}^{-j}B_{\mathfrak{d}}^{-\ell})\right|^2 \\ &+ \quad \sum_{j\geq 0}\sum_{\ell=\pm 2^j} \hat{f}(\xi)\left|\hat{\psi}(\xi A^{-j}B^{-\ell})\right|^2 \end{aligned}$$

converges in $\mathcal{S}'$. Since the equality is a straight consequence of (5.2), we will only show convergence in $\mathcal{S}'$ of the right-hand side of the equality for those shearlets with $j \geq 0$ (ψ_{-1} is in fact a scaling function of a Meyer wavelet). Suppose that $\hat{f}$ has order $\leq m$. This is, there exists an integer $n \geq 0$ and a constant C such that

$$\left|\langle \hat{f}, g\rangle\right| \leq C \sup_{|\alpha|\leq n, |\beta|\leq m} \|g\|_{\alpha,\beta}, \quad \text{for all } g \in \mathcal{S},$$

where $\|g\|_{\alpha,\beta} = \sup_{\xi\in\hat{\mathbb{R}}^2} |\xi^\alpha|\left|\partial^\beta g(\xi)\right|$ denotes the usual semi-norm in $\mathcal{S}$ for multi-indices α and β. Then,

$$\left|\langle \hat{f}\left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2, g\rangle\right| = \left|\langle \hat{f}, \left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2 g\rangle\right| \leq C \sup_{|\alpha|\leq n, |\beta|\leq m} \left\|\left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2 g\right\|_{\alpha,\beta}.$$

As in Lemma 2.5 in [16], one can prove that

$$\sup_{|\beta|=m} \left\|\partial^\beta \left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2\right\|_\infty \leq C 2^{-jm}.$$

Hence, by the compact support conditions of $(\psi_{A^{-j}B^{-\ell}})^\wedge(\xi)$ (see Subsection 2.3)

$$\begin{aligned}
&\sup_{|\alpha|\le n, |\beta|\le m} \left\| \left|(\psi_{A^{-j}B^{-\ell}})^\wedge\right|^2 g \right\|_{\alpha,\beta} \\
&\quad \le C \sup_{\xi\in\hat{\mathbb{R}}^2} \left[(1+|\xi|)^n \sup_{|\beta|\le m} \left| \partial^\beta \left|(\psi_{A^{-j}B^{-\ell}})^\wedge(\xi)\right|^2 \right| \sup_{|\beta|\le m} \left|\partial^\beta g(\xi)\right| \right] \\
&\quad \le C \sup_{\xi\in \operatorname{supp}(\psi_{A^{-j}B^{-\ell}})^\wedge(\xi)} (1+|\xi|)^n \sup_{|\beta|\le m} \left|\partial^\beta g(\xi)\right| \\
&\quad \le C \sup_{|\alpha|\le n+1, |\beta|\le m} \|g\|_{\alpha,\beta} \sup_{\xi\in \operatorname{supp}(\psi_{A^{-j}B^{-\ell}})^\wedge(\xi)} (1+|\xi|)^{-1} \\
&\quad \le C \sup_{|\alpha|\le n+1, |\beta|\le m} \|g\|_{\alpha,\beta} (1+2^{2j-4})^{-1} \le C 2^{-2j},
\end{aligned}$$

which proves the convergence in $\mathcal{S}'$.

■

Lemma 5.2.2. *Let $g \in \mathcal{S}'$ and $h \in \mathcal{S}$ be such that*

$$\operatorname{supp}\, \hat{g},\ \operatorname{supp}\, \hat{h} \subset [-1/2, 1/2]^2 B^\ell A^j = QB^\ell A^j, \qquad j \ge 0, \ell = -2^j, \dots, 2^j.$$

Then,

$$g * h = \sum_{k\in\mathbb{Z}^2} |\det\, A|^{-j} g(A^{-j}B^{-\ell}k) h(x - A^{-j}B^{-\ell}k),$$

with convergence in $\mathcal{S}'$.

Proof. Suppose first that $g \in \mathcal{S}$. We can express $\hat{g}$ by its Fourier series as

$$\hat{g}(\xi) = \sum_{k\in\mathbb{Z}^2} |\det\, A|^{-j/2} \mathbf{e}^{-2\pi i \xi A^{-j}B^{-\ell}k} \cdot \left(\int_{QB^\ell A^j} \hat{g}(\omega) \cdot |\det\, A|^{-j/2} \mathbf{e}^{2\pi i \omega A^{-j}B^{-\ell}k} d\omega \right).$$

By the Fourier inversion formula in $\hat{\mathbb{R}}^2$ we have

$$\hat{g}(\xi) = \sum_{k\in\mathbb{Z}^2} |\det\, A|^{-j/2} \mathbf{e}^{-2\pi i \xi A^{-j}B^{-\ell}k} \cdot g(A^{-j}B^{-\ell}k), \qquad \xi \in QB^\ell A^j.$$

Since $\hat{g}$ has compact support, $g(A^{-j}B^{-\ell}k)$ makes sense (by the Paley-Wiener theorem). Since supp $\hat{h} \subset QB^\ell A^j$ and $g * h = (\hat{g}\hat{h})^\vee$,

$$\begin{aligned}
g * h &= \sum_{k\in\mathbb{Z}^2} |\det\, A|^{-j} g(A^{-j}B^{-\ell}k) [\mathbf{e}^{-2\pi i \xi A^{-j}B^{-\ell}k} \hat{h}(\cdot)]^\vee \\
&= \sum_{k\in\mathbb{Z}^2} |\det\, A|^{-j} g(A^{-j}B^{-\ell}k) h(x - A^{-j}B^{-\ell}k),
\end{aligned}$$

which proves the convergence for $g \in \mathcal{S}$. To remove this assumption one uses the same standard regularization argument as in the proof of Lemma 8.2.3. This regularization argument is the same used in Lemma(6.10) in [14].

■

Theorem 5.2.3. *Let the shearlet system $\{\psi_{j,\ell,k}\}$ be constructed as in Subsection 5.1 such that it is a smooth Parseval frame that verifies (5.2). The composition of the*

analysis and synthesis operators $T_\psi \circ S_\psi$ (see (4.3) for the definitions) is the identity

$$f = \sum_{Q \in \mathcal{Q}_{AB}} \langle f, \psi_Q \rangle \psi_Q,$$

in $\mathcal{S}'$.

Proof. As in (3.2), $f * \tilde{\psi}_{A^{-j}B^{-\ell}}(A^{-j}B^{-\ell}k) = f * \tilde{\psi}_{A^{-j}B^{-\ell}}(x_P) = |\det A|^{j/2} \langle f, \psi_P \rangle$, where P is identified with (j, ℓ, k). Also, as in Subsection 3.1, $\psi_{A^{-j}B^{-\ell}}(x - A^{-j}B^{-\ell}k) = |\det A|^{j/2} \psi_P(x)$. Let $g = f * \tilde{\psi}_{A^{-j}B^{-\ell}}$ and $h = \psi_{A^{-j}B^{-\ell}}$. By construction, $\mathrm{supp}(\psi_{j,\ell,k})^\wedge(\xi) \subset QB^\ell A^j$. Therefore, Lemma 5.2.2 yields

$$\begin{aligned} f * \tilde{\psi}_{A^{-j}B^{-\ell}} * \psi_{A^{-j}B^{-\ell}} &= \sum_{k \in \mathbb{Z}^2} \langle f, \psi_{j,\ell,k} \rangle \psi_{j,\ell,k} \\ &= \sum_{P \in \mathcal{Q}^{j,\ell}} \langle f, \psi_P \rangle \psi_P. \end{aligned}$$

By appropriately summing over $\mathfrak{d} = 1, 2$, $j \geq 0$ and $\ell = -2^j, \ldots, 2^j$, Lemma 5.2.1 yields the result.

■

6. Relations between $\mathbf{F}_{p_1}^{\alpha_1,q_1}$ and $\mathbf{F}_{p_2}^{\alpha_2,q_2}(AB)$

In this section we prove embeddings of classical dyadic (isotropic) inhomogeneous Triebel-Lizorkin spaces into the just defined highly anisotropic inhomogeneous Triebel-Lizorkin spaces, and viceversa, for certain parameters. We also show that some functions in each of these spaces vanish in the other spaces for certain parameters. Let A and B be as in Subsection 2.3. A dyadic cube will be denoted by Q and a shear anisotropic "cube" (a parallelepiped) will be denoted by P.

We start with some definitions regarding the classical dyadic spaces (see Sections 2 and 12 in [13]). Let $\varphi, \theta, \Phi, \Theta$ be the analyzing and synthesizing functions of the φ-transform of Frazier and Jawerth. Then, φ, θ, Φ and Θ satisfy: 1) $\varphi, \theta, \Phi, \Theta \in \mathcal{S}$, 2) supp $\hat{\varphi}$, supp $\hat{\theta} \subset \{\xi \in \hat{\mathbb{R}}^2 : \frac{1}{2} \leq |\xi| \leq 2\}$ and supp $\hat{\Phi}$, supp $\hat{\Theta} \subset \{\xi \in \hat{\mathbb{R}}^2 : |\xi| \leq 2\}$, 3) $|\hat{\varphi}(\xi)|, \left|\hat{\theta}(\xi)\right| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$ and $\left|\hat{\Phi}(\xi)\right|, \left|\hat{\Theta}(\xi)\right| \geq c > 0$ if $|\xi| \leq \frac{5}{3}$, and 4) $\bar{\hat{\Phi}}(\xi)\hat{\Theta}(\xi) + \sum_{\nu \in \mathbb{Z}_+} \overline{\hat{\varphi}(2^{-\nu}\xi)} \hat{\theta}(2^{-\nu}\xi) = 1$. Let $\mathcal{D}_+$ denote the set of dyadic cubes with $l(Q) \leq 1$ where $l(Q)$ is the side size of Q. Let $\varphi_{\nu,k}(x) = 2^\nu \varphi(2^\nu x - k)$ be the L^2-normalized dilation and $\varphi_{2^\nu I}(x) = 2^{2\nu}\varphi(2^\nu x)$, where I is the identity matrix.

For $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p < \infty$, the (dyadic) inhomogeneous Triebel-Lizorkin space $\mathbf{F}_p^{\alpha,q}$ is the collection of all $f \in \mathcal{S}'$ such that (see Lemma 12.1 in [13] for a discussion on the dilation indices)

$$\|f\|_{\mathbf{F}_p^{\alpha,q}} = \|\Phi * f\|_{L^p} + \left\| \left(\sum_{\mathcal{D}_+} (2^{\nu\alpha} |\varphi_{2^\nu I} * f|)^q \right)^{1/q} \right\|_{L^p} < \infty.$$

For $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p < \infty$, the (dyadic) inhomogeneous Triebel-Lizorkin sequence space $\mathbf{f}_p^{\alpha,q}$ is the collection of all complex-valued sequences $\mathbf{s}$ such that

$$\|\mathbf{s}\|_{\mathbf{f}_p^{\alpha,q}} = \left\| \left(\sum_{Q:l(Q)\leq 1} (2^{\nu\alpha} |s_Q| \, \tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^p} < \infty,$$

where $\tilde{\chi}_Q(x) = |Q|^{-\frac{1}{2}} \chi_Q(x)$ is the L^2-normalized characteristic function of $Q \in \mathcal{D}_+$.

Let $\mathbf{s} = \{s_Q\}_Q$, where we identify Q with the pair $(\nu, k) \in \mathbb{Z}_+ \times \mathbb{Z}^2$. For $0 < r \leq \infty$ and $\lambda > 0$, define the sequence $\mathbf{s}^*_{r,\lambda} = \{(s^*_{r,\lambda})_Q\}_{Q \in \mathcal{D}_+}$ by

$$(s^*_{r,\lambda})_{Q'} = \left(\sum_{Q:l(Q)=l(Q')} \frac{|s_Q|^r}{(1 + l(Q')^{-1} |x_Q - x_{Q'}|)^\lambda} \right)^{1/r},$$

where $x_Q = 2^{-\nu}k$ is the lower left corner of $Q_{\nu,k} = 2^{-\nu}(Q_0 + k)$, see p. 48 of [13] and compare with the similar definition at Subsection 4.2.

6.1. **The embeddings.** We start with a result on almost orthogonality of functions under highly anisotropic and dyadic dilations.

Lemma 6.1.1. *Let $\psi, \varphi \in \mathcal{S}$. For $j \geq 0$, $|\ell| \leq 2^j$ and $k \in \mathbb{Z}^2$,*

$$\int_{\mathbb{R}^2} \left|\psi(B^\ell A^j (x-y))\right| \left|\varphi(2^{2j} y)\right| dy \leq \frac{2^{-3j}}{(1 + 2^j |x|)^N},$$

for all $N > 2$.

Proof. Since $\psi, \varphi \in \mathcal{S}$,

$$\int_{\mathbb{R}^2} \left|\psi(B^\ell A^j (x-y))\right| \left|\varphi(2^{2j} y)\right| dy \lesssim \int_{\mathbb{R}^2} \frac{1}{(1 + |B^\ell A^j (x-y)|)^N} \frac{1}{(1 + |2^{2j} y|)^N} dy.$$

Define

$$\begin{aligned} E_1 &= \{y \in \mathbb{R}^2 : 2^j |x-y| \leq 3\} \\ E_2 &= \{y \in \mathbb{R}^2 : 2^j |x-y| > 3, |y| \leq |x|/2\} \\ E_3 &= \{y \in \mathbb{R}^2 : 2^j |x-y| > 3, |y| > |x|/2\}. \end{aligned}$$

For $y \in E_1$, $1 + 2^j |x| \leq 1 + 2^j |x-y| + 2^j |y| \leq 4(1 + 2^{2j} |y|)$. If $y \in E_3$, $1 + 2^j |x| \leq 1 + 2^{j+1} |y| \leq 2(1 + 2^{2j} |y|)$. When $y \in E_2$, $2^{j-1} |x| < 2^j(|x| - |y|) \leq 2^j |x-y|$, which implies $4\left|B^\ell A^j (x-y)\right| \geq 2^{j-1} |x-y| + 3 \cdot 2^{j-1} |x-y| \geq \frac{3}{2} + \frac{3}{2} 2^{j-1} |x|$ or $8(1 + \left|B^\ell A^j (x-y)\right|) \geq 1 + 2^j |x|$. Hence,

$$\begin{aligned} &\int_{\mathbb{R}^2} \left|\psi(B^\ell A^j (x-y))\right| \left|\varphi(2^{2j} y)\right| dy \\ &\lesssim \frac{1}{(1 + 2^j |x|)^N} \int_{E_1 \cup E_3} \frac{1}{(1 + |B^\ell A^j (x-y)|)^N} dy \\ &\quad + \frac{1}{(1 + 2^j |x|)^N} \int_{E_2} \frac{1}{(1 + 2^{2j} |y|)^N} dy \end{aligned}$$

$$\lesssim \left[\frac{2^{-3j}}{(1+2^j|x|)^N} + \frac{2^{-4j}}{(1+2^j|x|)^N}\right] \lesssim \frac{2^{-3j}}{(1+2^j|x|)^N},$$

for all $N > 2$.

■

The definitions of E_1, E_2, E_3 in Lema 6.1.1 allow us to have a “height” of 2^{-3j} and a decreasing of $(1+2^j|x|)^{-N}$. By defining E_1, E_2, E_3 as in Lemma 8.1.2 would only yield a “height” of 2^{-2j} and a decreasing of $(1+|x|)^{-N}$.

Theorem 6.1.2. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p < \infty$ and $\lambda > 2\max(1, 1/q, 1/p)$. If $3\alpha_2 + \frac{1}{q} + \lambda < \alpha_1$,*

$$\mathbf{F}_p^{\alpha_1,q} \hookrightarrow \mathbf{F}_p^{\alpha_2,q}(AB).$$

Proof. To shorten notation write $\mathbf{F}_1 = \mathbf{F}_p^{\alpha_1,q}$, $\mathbf{f}_1 = \mathbf{f}_p^{\alpha_1,q}$ and $\mathbf{F}_2 = \mathbf{F}_p^{\alpha_2,q}(AB)$. We will actually prove that, for $f = \sum_{Q\in\mathcal{D}_+} s_Q\varphi_Q \in \mathbf{F}_1$,

$$\|f\|_{\mathbf{F}_2} \lesssim \left\|\mathbf{s}^*_{1,\lambda}\right\|_{\mathbf{f}_1} \lesssim \|\mathbf{s}\|_{\mathbf{f}_1} \lesssim \|f\|_{\mathbf{F}_1},$$

where, of course, the inequality we are interested to prove is the first one and the last two are proved in [13]. From the compact support conditions of $(\varphi_{\nu,k})^\wedge$ and $(\psi_{A^{-j}B^{-\ell}})^\wedge$ and their dyadic and highly anisotropic expansion, respectively, we formally get

$$\psi_{A^{-j}B^{-\ell}} * f = \sum_{\nu=2j-5}^{2j} \sum_{k\in\mathbb{Z}^2} s_{\nu,k}\psi_{A^{-j}B^{-\ell}} * \varphi_{\nu,k}.$$

Therefore, writing $\psi_{A^{-j}B^{-\ell}}(x) = |\det A|^j \psi(B^\ell A^j x)$ and $\varphi_{\nu,k}(x) = 2^\nu\varphi(2^\nu x - k)$, Lemma 6.1.1 yields

$$\begin{aligned}
\|f\|_{\mathbf{F}_2} &= \left\| \left(\sum_{j\geq 0} 2^{3j\alpha_2 q} \sum_{|\ell|\leq 2^j} [\left| \sum_{\nu=2j-5}^{2j} \sum_{k\in\mathbb{Z}^2} s_{\nu,k}\psi_{A^{-j}B^{-\ell}} * \varphi_{\nu,k}(\cdot)\right|]^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left(\sum_{j\geq 0} 2^{3j\alpha_2 q} \sum_{|\ell|\leq 2^j} [\left| \sum_{k\in\mathbb{Z}^2} s_{2j,k}\psi_{A^{-j}B^{-\ell}} * \varphi_{2j,k}(\cdot)\right|]^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left(\sum_{j\geq 0} 2^{3j\alpha_2 q} \sum_{|\ell|\leq 2^j} [\sum_{k\in\mathbb{Z}^2} |s_{2j,k}| \frac{2^{2j}}{(1+2^j|\cdot + 2^{-2j}k|)^N}]^q \right)^{1/q} \right\|_{L^p},
\end{aligned}$$

for all $N > 2$. Let $\lambda > 2\max(1, r/q, r/p)$ for some $r > 0$. Following the proof of the second part of Theorem 4.3.1, if $x \in Q'$ and $Q' \in \mathcal{D}^{2j}$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^2} \frac{|s_{2j,k}|\, 2^{2j}}{(1+2^j|x-2^{-2j}k|)^\lambda} \\
&\quad= \sum_{k\in\mathbb{Z}^2} \frac{|s_{2j,k}|\, 2^{2j}\cdot 2^{j\lambda}}{2^{j\lambda}(1+2^j|x-2^{-2j}k|)^\lambda} \leq 2^{j\lambda} \sum_{k\in\mathbb{Z}^2} \frac{|s_{2j,k}|\, 2^{2j}}{(1+2^{2j}|x-2^{-2j}k|)^\lambda} \\
&\quad\lesssim 2^{j\lambda} \sum_{Q\in\mathcal{D}^{2j}} |Q|^{-\frac{1}{2}} \left|(s^*_{1,\lambda})_Q\right| \chi_Q(x) = 2^{j\lambda} \sum_{Q\in\mathcal{D}^{2j}} \left|(s^*_{1,\lambda})_Q\right| \tilde{\chi}_Q(x),
\end{aligned}$$

since $\mathcal{D}^{2j}$ is a partition of $\mathbb{R}^2$. Hence,

$$\begin{aligned}
\|f\|_{\mathbf{F}_2} &\lesssim \left\| \left(\sum_{j\geq 0} 2^{3j\alpha_2 q}(2^{j+1}+1)[2^{j\lambda} \sum_{Q\in\mathcal{D}^{2j}} \left|(s^*_{1,\lambda})_Q\right| \tilde{\chi}_Q(\cdot)]^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left(\sum_{j\geq 0} \sum_{Q\in\mathcal{D}^{2j}} [2^{3j\alpha_2+\frac{j}{q}+j\lambda} \left|(s^*_{1,\lambda})_Q\right| \tilde{\chi}_Q(\cdot)]^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left(\sum_{j\geq 0} \sum_{Q\in\mathcal{D}^{j}} [2^{3j\alpha_2+\frac{j}{q}+j\lambda} \left|(s^*_{1,\lambda})_Q\right| \tilde{\chi}_Q(\cdot)]^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left(\sum_{Q\in\mathcal{D}_+} [|Q|^{-\frac{\alpha_1}{2}} \left|(s^*_{1,\lambda})_Q\right| \tilde{\chi}_Q(\cdot)]^q \right)^{1/q} \right\|_{L^p} = \left\|\mathbf{s}^*_{1,\lambda}\right\|_{\mathbf{f}_1},
\end{aligned}$$

since $2\mathbb{N} \subset \mathbb{N}$ and $2^{3j\alpha_2+\frac{j}{q}+j\lambda} \leq 2^{j\alpha_1} = |Q|^{-\frac{\alpha_1}{2}}$ for a $Q \in \mathcal{D}^j$. Let now $r = \min(q,p)$. Since $\lambda/2 > \max(1, r/q, r/p)$, $r/(\lambda/2) < \min(r,q,p)$. We can choose a such that $r/(\lambda/2) < a < \min(r,q,p)$. Then, $0 < a < r < \infty$, $\lambda > 2r/a$, $q/a > 1$ and $p/a > 1$. Following the proof of Lemma 2.3 in [13] (in which a and λ are defined differently) we get $\left\|\mathbf{s}^*_{1,\lambda}\right\|_{\mathbf{f}_1} \lesssim \|\mathbf{s}\|_{\mathbf{f}_1}$, and from Theorem 2.2 in [13], $\|\mathbf{s}\|_{\mathbf{f}_1} \lesssim \|f\|_{\mathbf{F}_1}$, which finishes the proof. ■

Theorem 6.1.3. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $0 < q \leq \infty$, and $0 < p < \infty$. If $\alpha_1 + 1 \leq 3\alpha_2$,*

$$\mathbf{F}^{\alpha_2,q}_p(AB) \hookrightarrow \mathbf{F}^{\alpha_1,q}_p.$$

Proof. To shorten notation write $\mathbf{F}_1 = \mathbf{F}^{\alpha_1,q}_p$, $\mathbf{F}_2 = \mathbf{F}^{\alpha_2,q}_p(AB)$ and $\mathbf{f}_2 = \mathbf{f}^{\alpha_2,q}_p$. Suppose $f = \sum_{P\in\mathcal{Q}_{AB}} s_P\psi_P \in \mathbf{F}_2$ and let $\lambda > 3\max(1, 1/q, 1/p)$. From the compact support conditions on $(\varphi_{2^\nu I})^\wedge$ and $(\psi_{j,\ell,k})^\wedge$, $j \sim \lfloor \nu/2 \rfloor$. Also, for $\nu \geq 0$, $\lfloor \nu/2 \rfloor$ runs through $\mathbb{Z}_+$ twice. So, writing $\psi_{\nu,\ell,k}(x) = |\det A|^{\nu/2}\, \psi(B^\ell A^\nu x - k)$ and $\varphi_{2^{2\nu} I}(x) = 2^{4\nu}\varphi(2^{2\nu}x)$,

$$\begin{aligned}
\|f\|_{\mathbf{F}_1} &= \left\| \left(\sum_{\nu\geq 0} (|Q_\nu|^{-\alpha_1/2} \left|\varphi_{2^\nu I} * f(\cdot)\right|)^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim \left\| \left(\sum_{\nu\geq 0} 2^{\nu\alpha_1 q} (\left| \sum_{|\ell|\leq 2^{\lfloor \nu/2 \rfloor}} \sum_{k\in\mathbb{Z}^2} s_{\lfloor \nu/2 \rfloor,\ell,k} \cdot \varphi_{2^\nu I} * \psi_{\lfloor \nu/2 \rfloor,\ell,k}(\cdot) \right|)^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim 2 \left\| \left(\sum_{\nu\geq 0} 2^{\nu\alpha_1 q} (\left| \sum_{|\ell|\leq 2^{\nu}} \sum_{k\in\mathbb{Z}^2} s_{\nu,\ell,k} \cdot \varphi_{2^{2\nu} I} * \psi_{\nu,\ell,k}(\cdot) \right|)^q \right)^{1/q} \right\|_{L^p}
\end{aligned}$$

$$\lesssim \left\| \left(\sum_{\nu \geq 0} 2^{\nu\alpha_1 q} (\sum_{|\ell| \leq 2^\nu} \sum_{k \in \mathbb{Z}^2} |s_{\nu,\ell,k}| \cdot \frac{2^{-3\nu} \cdot 2^{4\nu} \cdot 2^{3\nu/2}}{(1+2^\nu |\cdot - A^{-\nu}B^{-\ell}k|)^N})^q \right)^{1/q} \right\|_{L^p},$$

for all $N > 2$, by Lemma 6.1.1. Continuing as in the second part of the proof of Theorem 4.3.1, if $x \in P'$ and $P' \in \mathcal{Q}^{\nu,\ell}$,

$$\begin{aligned} \|f\|_{\mathbf{F}_1} &\lesssim \left\| \left(\sum_{\nu \geq 0} 2^{\nu\alpha_1 q + \nu q} (\sum_{|\ell| \leq 2^\nu} \sum_{P \in \mathcal{Q}^{\nu,\ell}} |s_P| \cdot \frac{|P|^{-1/2}}{(1+2^\nu |\cdot - x_P|)^N})^q \right)^{1/q} \right\|_{L^p} \\ &\lesssim \left\| \left(\sum_{\nu \geq 0} 2^{\nu q(\alpha_1+1)} [\sum_{|\ell| \leq 2^\nu} \sum_{P \in \mathcal{Q}^{\nu,\ell}} (s^*_{1,N})_P \tilde{\chi}_P(\cdot)]^q \right)^{1/q} \right\|_{L^p}, \end{aligned}$$

because $\mathcal{Q}^{\nu,\ell}$ is a partition of $\mathbb{R}^2$. However, at this point we cannot use the "partition of $\mathbb{R}^2$" on $\sum_{|\ell| \leq 2^\nu} \sum_{P \in \mathcal{Q}^{\nu,\ell}} \tilde{\chi}_P$. Therefore, if $0 < q \leq 1$ we use the q-triangle inequality $|a+b|^q \leq |a|^q + |b|^q$ ($N > 2/q$) or Hölder's inequality if $1 < q$ ($N > 2$) to get (by hypothesis $\lambda > 3\max(1, 1/q, 1/p) > N$)

$$\begin{aligned} \|f\|_{\mathbf{F}_1} &\lesssim \left\| \left(\sum_{P \in \mathcal{Q}_{AB}} [2^{\nu(\alpha_1+1)} \left|(s^*_{1,\lambda})_P\right| \tilde{\chi}_P(\cdot)]^q \right)^{1/q} \right\|_{L^p} \\ &\lesssim \left\| \left(\sum_{P \in \mathcal{Q}_{AB}} [|P|^{-\alpha_2} \left|(s^*_{1,\lambda})_P\right| \tilde{\chi}_P(\cdot)]^q \right)^{1/q} \right\|_{L^p} = \left\|\mathbf{s}^*_{1,\lambda}\right\|_{\mathbf{f}_2}. \end{aligned}$$

By Lemma 4.2.4 and Theorem 4.3.1 the proof is complete.

■

6.2. **Further relations.** A dyadic cube at scale ν will be denoted by Q_ν and a shear anisotropic "cube" (a parallelepiped) at scale j will be denoted by P_j.

Theorem 6.2.1. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $0 < q_1, q_2 \leq \infty$ and $0 < p_1, p_2 < \infty$. Then, there exist sequences of functions $\{f^{(j)}\}_{j \geq 0}$ such that $\left\|f^{(j)}\right\|_{\mathbf{F}^{\alpha_2,q_2}_{p_2}(AB)} \approx 1$, but that $\left\|f^{(j)}\right\|_{\mathbf{F}^{\alpha_1,q_1}_{p_1}} \to 0$, $j \to \infty$, when $3(\alpha_2 - 1/p_2) > 2\alpha_1 - 1/p_1 + 1$.*

Proof. For a sequence $\mathbf{s}^{(j)} = \{s_{j,0,0}\}_{j \geq 0}$ such that $|s_{j,0,0}| = |P_j|^{\alpha_2 - \frac{1}{p_2} + \frac{1}{2}}$, we have $\left\|\mathbf{s}^{(j)}\right\|_{\mathbf{f}^{\alpha_2,q_2}_{p_2}(AB)} = 1$, for all $j \geq 0$. Thus, $f^{(j)}(x) = s_{j,0,0}\psi_{j,0,0}(x) \in \mathbf{F}^{\alpha_2,q_2}_{p_2}(AB)$ with $\left\|f^{(j)}\right\|_{\mathbf{F}^{\alpha_2,q_2}_{p_2}(AB)} \approx 1$. From the compact support conditions on $\hat{\varphi}$ y $\hat{\psi}$ the support of $(\psi_{j,0,0})^\wedge$ overlaps with the support of $(\varphi_{\nu,0})^\wedge$ only when $2j - 5 \leq \nu < 2j$. Therefore, since assuming $\nu = 2j$ and $\left|f^{(j)} * \varphi_{2^{2j}I}\right| = 2^{2j}\left|f^{(j)} * \varphi_{\nu,0}\right|$, Lemma 6.1.1 yields

$$\begin{aligned} \left|f^{(j)} * \varphi_{2^\nu I}(x)\right| &= \left| \int_{\mathbb{R}^2} |P_j|^{\alpha_2 - \frac{1}{p_2} + \frac{1}{2}} |\det A|^{j/2} \psi(A^j(x-y)) 2^{4j} \varphi(2^{2j}y) dy \right| \\ &\lesssim 2^{-3j(\alpha_2 - \frac{1}{p_2} + \frac{1}{2}) + \frac{3j}{2} + 4j} \int_{\mathbb{R}^2} \left|\psi(A^j(x-y))\right| \left|\varphi(2^{2j}y)\right| dy \end{aligned}$$

$$\lesssim \frac{2^{-3j(\alpha_2-\frac{1}{p_2})+j}}{(1+2^j|x|)^N},$$

for every $N>2$. Then, for $j>0$ large enough and for N such that $Np_1>2$, we have

$$\begin{aligned}
\left\|f^{(j)}\right\|_{\mathbf{F}_{p_1}^{\alpha_1,q_1}} &= \left\|\left(\sum_{\nu=2j-5}^{2j-1}[2^{\nu\alpha_1}\left|f^{(j)}*\varphi_{2^\nu I}\right|]^{q_1}\right)^{1/q_1}\right\|_{L^{p_1}} \\
&\leq C_{N,q_1}\left(\int_{\mathbb{R}^2} 2^{2j\alpha_1 p_1}\cdot\frac{[2^{-3j(\alpha_2-\frac{1}{p_2})+j}]^{p_1}}{(1+2^j|x|)^{Np_1}}dx\right)^{1/p_1} \\
&= C_{N,q_1}2^{2j\alpha_1-3j(\alpha_2-\frac{1}{p_2})+j-\frac{j}{p_1}},
\end{aligned}$$

which tends to 0 as $j\to\infty$ if $2\alpha_1-\frac{1}{p_1}+1<3(\alpha_2-\frac{1}{p_2})$.

■

Theorem 6.2.2. *Let $\alpha_1,\alpha_2\in\mathbb{R}$, $0<q_1,q_2\leq\infty$ and $0<p_1,p_2<\infty$. Then, there exist sequences of functions $\{f^{(\nu)}\}_{\nu\geq 0}$ such that $\left\|f^{(\nu)}\right\|_{\mathbf{F}_{p_1}^{\alpha_1,q_1}}\approx 1$, but that $\left\|f^{(\nu)}\right\|_{\mathbf{F}_{p_2}^{\alpha_2,q_2}(AB)}\to 0$, $\nu\to\infty$, when $2\alpha_1-4/p_1>3\alpha_2+1/q_2-1/p_2$.*

Proof. For a sequence $\mathbf{s}^{(\nu)}=\{s_{\nu,0}\}_{j\geq 0}$ such that $|s_{\nu,0}|=|Q_\nu|^{\frac{\alpha_1}{2}-\frac{1}{p_1}+\frac{1}{2}}$, $\left\|\mathbf{s}^{(\nu)}\right\|_{\mathbf{f}_{p_1}^{\alpha_1,q_1}}=1$, for all $\nu\geq 0$. This means that $f^{(\nu)}(x)=s_{\nu,0}\varphi_{\nu,0}(x)\in\mathbf{F}_{p_1}^{\alpha_1,q_1}$ and $\left\|f^{(\nu)}\right\|_{\mathbf{F}_{p_1}^{\alpha_1,q_1}}\approx 1$, for all $\nu\geq 0$. With the same arguments on the conditions of the support of $(\varphi_{\nu,0})^\wedge$ and $(\psi_{j,\ell,0})^\wedge$ we assume $\nu=2j$ to get

$$\begin{aligned}
\left|f^{(2j)}*\psi_{A^{-j}B^{-\ell}}(x)\right| &= \left|\int_{\mathbb{R}^2} s_{2j,0}2^{2j}\varphi(2^{2j}y)\left|\det A\right|^j\psi(B^\ell A^j(x-y))dy\right| \\
&\lesssim 2^{-4j(\frac{\alpha_1}{2}-\frac{1}{p_1}+\frac{1}{2})+2j+3j}\int_{\mathbb{R}^2}\left|\psi(B^\ell A^j(x-y))\right|\left|\varphi(2^{2j}y)\right|dy \\
&\lesssim \frac{2^{-4j(\frac{\alpha_1}{2}-\frac{1}{p_1})}}{(1+2^j|x|)^N},
\end{aligned}$$

for every $N>2$, by Lemma 6.1.1. Then, for $\nu>0$ large enough and $N>2$ such that $Np_2>2$,

$$\begin{aligned}
\left\|f^{(2j)}\right\|_{\mathbf{F}_2} &\lesssim \left\|\left(\sum_{|\ell|\leq 2^j}[|P_j|^{-\alpha_2}\left|f^{2j}*\psi_{A^{-j}B^{-\ell}}\right|]^{q_2}\right)^{1/q_2}\right\|_{L^{p_2}} \\
&\lesssim \left(\int_{\mathbb{R}^2}(\sum_{|\ell|\leq 2^j}[2^{3j\alpha_2}\frac{2^{-4j(\frac{\alpha_1}{2}-\frac{1}{p_1})}}{(1+2^j|x|)^N}]^{q_2})^{p_2/q_2}dx\right)^{1/p_2} \\
&= \left(\int_{\mathbb{R}^2}((2^{j+1}+1)[\frac{2^{j(3\alpha_2-4(\frac{\alpha_1}{2}-\frac{1}{p_1}))}}{(1+2^j|x|)^N}]^{q_2})^{p_2/q_2}dx\right)^{1/p_2} \\
&\leq 2^{j(3\alpha_2-4(\frac{\alpha_1}{2}-\frac{1}{p_1})+\frac{1}{q_2})}\left(\int_{\mathbb{R}^2}\frac{1}{(1+2^j|x|)^{Np_2}}dx\right)^{1/p_2}
\end{aligned}$$

$$\lesssim \ 2^{j(3\alpha_2 - 4(\frac{\alpha_1}{2} - \frac{1}{p_1}) + \frac{1}{q_2})} \cdot 2^{-\frac{j}{p_2}},$$

which tends to 0, as $j \to \infty$, if $3\alpha_2 - 4(\frac{\alpha_1}{2} - \frac{1}{p_1}) + \frac{1}{q_2} - \frac{1}{p_2} < 0$.

■

7. Weights

To extend this work to the weighted case $w \in A_\infty = \cup_{p>1} A_p$, one can follow [6] and [5]. The spaces $\mathbf{F}_p^{\alpha,q}(AB)$ and $\mathbf{f}_p^{\alpha,q}(AB)$ are then defined by $L^p(w)$ quasi-norms. For the weighted version of the Fefferman-Stein inequality we refer the reader to [1] or to Remark 6.5 of Chapter V in [15]. One adds $w \in A_{p_0}$ to the statement of Lemma 4.2.4 and N is chosen so that $N > 3\max(1, r/q, rp_0/p)$. Regarding the proof of Lemma 4.2.4, λ should be chosen so that $N > 3\lambda/2 > 3\max(1, r/q, rp_0/p)$. For Theorem 4.3.1 one adds $w \in A_\infty$. For its proof, λ should be chosen such that $0 < \lambda < \min(p/p_0, q)$ and N such that $N > 3\max(1, 1/q, p_0/p)$.

8. Proofs

We prove results of Subsections 3.1 and 4.2 in each of the Subsections 8.s, $s = 1, 2$.

8.1. **Proofs for Subsection 3.1.** In order to prove Lemma 3.1.1 we need two previous results. The first one establishes that the sheared ellipsoids, in a certain scale j and for all shear parameter $\ell = -2^j, \dots, 2^j$, contain the circle in a lower scale $j-1$. This allows us to bound from below the distance of a point $y \in \mathbb{R}^2$ under the $B^\ell A^j$ operation. The second one is a basic "almost orthogonality" result from which we derive Lemma 3.1.1.

Lemma 8.1.1. *Let A and B be as in Section 3. Then,*

$$2^{j-1}|x| < 2^{j-\frac{1}{2}}|x| < \left|B^\ell A^j x\right|,$$

for all $j \geq 0$, $\ell = -2^j, \dots, 2^j$ and all $x \in \mathbb{R}^2$.

Proof. For any $x \in \mathbb{R}^2$,

$$B^\ell A^j x = \begin{pmatrix} 2^{2j}x_1 + 2^j \ell x_2 \\ 2^j x_2. \end{pmatrix}$$

With out loss of generality we can take $x \in \partial\mathbb{U}$, where $\mathbb{U}$ is the unit disk. Then, $2^{j-1}|x| = 2^{j-1}$, and

$$\begin{aligned} \left|B^\ell A^j x\right| &= ((2^{2j}x_1 + 2^j \ell x_2)^2 + (2^j x_2)^2)^{1/2} \\ &= (2^{4j}x_1^2 + 2 \cdot 2^{3j}\ell x_1 x_2 + 2^{2j}\ell^2 x_2^2 + (2^j x_2)^2)^{1/2} \\ &= (2^{4j}x_1^2 + 2 \cdot 2^{3j}\ell x_1(1-x_1^2)^{1/2} + 2^{2j}\ell^2(1-x_1^2) + 2^{2j}(1-x_1^2))^{1/2}. \end{aligned}$$

The extrema of $x_1(1-x_1^2)^{1/2} = \pm 1/2$ take place in $x_1 = \pm 1/\sqrt{2}$ and $-2^{3j}\ell + 2^{2j-1}\ell^2$ is minimized at $\ell = 2^j$. So,

$$\left|B^\ell A^j x\right| \geq (2^{4j-1} - 2^{3j}\ell + 2^{2j-1}\ell^2 + 2^{2j-1})^{1/2} \geq 2^{j-\frac{1}{2}}.$$

■

Similarly one can also prove that, for $i \geq j$, $\left|B^m A^i A^{-j} B^{-\ell} x\right| \geq 2^{i-j-1}|x|$ holds true. To see this observe that for $x \in \mathbb{U}$,

$$\begin{aligned}\left|B^m A^i A^{-j} B^{-\ell} x\right| &= (2^{4(i-j)} x_1^2 \pm 2^{2(i-j)+1}(\ell 2^{2(i-j)} + m 2^{i-j}) x_1 (1-x_1^2)^{1/2} \\ &\quad +(\ell 2^{2(i-j)} + m 2^{i-j})^2 (1-x_1^2) + 2^{2(i-j)}(1-x_1^2))^{1/2}.\end{aligned}$$

With the same extrema $x_1(1-x_1^2)^{1/2} = \pm 1/2$ as before and minimizing $-2^{2(i-j)}a + a^2/2$, where $a = \ell 2^{2(i-j)} + m 2^{i-j}$, the result follows as in the proof of Lemma 8.1.1.

Observe that this result cannot be applied to the case of the "discrete shearlets", since the shear parameter in this case runs through $\mathbb{Z}$ making the ellipsoids thinner as $\ell \to \pm\infty$ and consequently intersect the inner circle.

The next result is fundamental to prove our first "almost orthogonality" result in form of a convolution in the space domain (see (3.2)).

Lemma 8.1.2. *Let $g, h \in \mathcal{S}$. Then, for all $N > 2$ and $i = j-1, j, j+1 \geq 0$, there exists $C_N > 0$ such that*

$$|g_{j,\ell,k} * h_{i,m,n}(x)| \leq \frac{C_N}{(1 + 2^j |x - A^{-i}B^{-m}n - A^{-j}B^{-\ell}k|)^N},$$

for all $x \in \mathbb{R}^2$.

Proof. First, since $g, h \in \mathcal{S}$ then $|g(x)|, |h(x)| \leq \frac{C_N}{(1+|x|)^N}$ for all $N \in \mathbb{N}$. With the notation for $\varphi_M(x)$ we have

$$\begin{aligned} g_{j,\ell,k} * h_{i,m,n}(x) &= \int_{\mathbb{R}^2} 2^{3j/2} g(B^\ell A^j y - k) \cdot 2^{3i/2} h(B^m A^i (x-y) - n) dy \\ &= 2^{-3(i-j)/2} g_{0,0,0} * h_{B^\ell A^j A^{-i} B^{-m}}(x'),\end{aligned}$$

with $x' = B^\ell A^j x - k - B^\ell A^j A^{-i} B^{-m} n = B^\ell A^j (x - A^{-j}B^{-\ell}k - A^{-i}B^{-m}n)$. Following [19, §6], define

$$\begin{aligned} E_1 &= \{y \in \mathbb{R}^2 : |x' - y| \leq 3\} \\ E_2 &= \{y \in \mathbb{R}^2 : |x' - y| > 3, |y| \leq |x'|/2\} \\ E_3 &= \{y \in \mathbb{R}^2 : |x' - y| > 3, |y| > |x'|/2\}.\end{aligned}$$

For $y \in E_1$ we have $1 + |x'| \leq 1 + |x' - y| + |y| \leq 4(1 + |y|)$. If $y \in E_3$, $1 + |x'| \leq 1 + 2|y| \leq 2(1 + |y|)$. When $y \in E_2$, $\frac{1}{2}|x'| \leq |x'| - |y| < |x' - y|$, which implies $4|x' - y| = |x' - y| + 3|x' - y| \geq 3 + \frac{3}{2}|x'| \geq 1 + |x'|$ and $1 + \left|B^m A^i A^{-j} B^{-\ell}(x' - y)\right| \geq 2^{i-j-1}|x' - y| \geq 2^{i-j-3}(1 + |x'|)$ (see Lemma 8.1.1). Thus,

$$\begin{aligned} |g_{j,\ell,k} * h_{i,m,n}(x')| &\lesssim \left\{\int_{E_1 \cup E_3} + \int_{E_2}\right\} \frac{2^{-3(i-j)/2}}{(1+|y|)^N} \frac{\left|\det B^m A^i A^{-j} B^{-\ell}\right|}{(1 + |B^m A^i A^{-j} B^{-\ell}(x' - y)|)^N} dy \\ &\lesssim \frac{2^{-3(i-j)/2}}{(1+|x'|)^N} \int_{E_1 \cup E_3} \frac{\left|\det B^m A^i A^{-j} B^{-\ell}\right|}{(1 + |B^m A^i A^{-j} B^{-\ell}(x' - y)|)^N} dy \\ &\quad + \frac{2^{-3(i-j)/2} 2^{3(i-j)}}{2^{(i-j-3)(N)}(1+|x'|)^N} \int_{E_2} \frac{1}{(1+|y|)^N} dy \\ &\lesssim \frac{C_N 2^{-3(i-j)/2}}{(1+|x'|)^N} \int_{\mathbb{R}^2} \frac{1}{(1+|y|)^N} dy \end{aligned}$$

$$+\frac{C_N 2^{3(i-j)/2}}{2^{(i-j-3)(N)}(1+|x'|)^N}\int_{\mathbb{R}^2}\frac{1}{(1+|y|)^N}dy,$$

for some $C_N > 0$ and all $N > 2$. The result follows from the fact that $|i-j| \leq 1$ and by replacing back $x' = B^\ell A^j(x - A^{-j}B^{-\ell}k - A^{-i}B^{-m}n)$ in the estimates above and because $\left|B^\ell A^j y\right| > 2^{j-1}|y|$ (see Lemma 8.1.1) implies that

$$\frac{1}{(1+|B^\ell A^j y|)^N} < \frac{1}{(1+2^{j-1}|y|)^N} < \frac{2^N}{(1+2^j|y|)^N}.$$

■

As a corollary for Lemma 8.1.2 we have our first "almost orthogonality" property for the anisotropic and shear operations for functions in $\mathcal{S}$.

Proof of Lemma 3.1.1. Identify $(j,\ell,0)$ with P and (i,m,n) with Q. Write $|g_{A^{-j}B^{-\ell}} * h_{i,m,n}(x)| = \left||P|^{-1/2} g_P * h_Q\right|$. Since $|i-j| \leq 1$, $|P|^{-1/2} \sim |Q|^{-1/2}$. Then, Lemma 8.1.2 yields

$$\left||P|^{-1/2} g_P * h_Q\right| \leq \frac{C_N |P|^{-1/2}}{(1+2^j|x-x_Q|)^N} \lesssim \frac{C_N |Q|^{-1/2}}{(1+2^j|x-x_Q|)^N}.$$

■

Our second "almost orthogonality" result is stated in the Fourier domain and gives more information since this time we take into account the shear parameter ℓ.

Proof of Lemma 3.1.2. This is a direct consequence of the construction and dilation of the shearlets. Since k and n are translation parameters they do not seem reflected in the support of $(\psi_{j,\ell,k})^\wedge$ or $(\psi_{i,m,n})^\wedge$. By construction and by (2.3) one scale j intersects with scales $j-1$ and $j+1$, only.

1) For one fixed scale j and by (2.4) there exist **2** overlaps at the same scale j: one with $(\psi_{j,\ell-1,k'})^\wedge$ and other with $(\psi_{j,\ell+1,k''})^\wedge$ for all $k', k'' \in \mathbb{Z}^2$.

2) Regarding scale $j-1$, one fixed $(\psi_{j,\ell,k})^\wedge$ overlaps with **3** other shearlets $(\psi_{j-1,m,k'})^\wedge$ at most for all $k, k' \in \mathbb{Z}^2$ because of 1) and because the supports of the shearlets at scale $j-1$ have larger width than those of scale j.

3) For a fixed scale j consider the next three regions: supp $(\psi_{j,\ell-1,k})^\wedge \cap$supp $(\psi_{j,\ell,k'})^\wedge = R_{-1}$, supp $(\psi_{j,\ell,k'})^\wedge \cap$ supp $(\psi_{j,\ell+1,k''})^\wedge = R_{+1}$ and supp $(\psi_{j,\ell,k'})^\wedge \setminus (R_{-1} \cup R_{+1}) = R_0$. Again by construction, there can only be two overlaps for each ξ at any scale. Then, there exist at most two shearlets at scale $j+1$ that overlap with each of the three regions R_i, $i = -1, 0, +1$ in scale j: an aggregate of **6** for all translation parameters $k, k', k'' \in \mathbb{Z}^2$ at any scale j or $j+1$.

Summing the number of overlaps at each scale gives the result.

■

8.2. **Proofs of Subsection 4.2.** To prove our results we follow [13], [19, §6.3], [5] and [24, §1.3]. Some previous well known definitions and results are necessary.

Definition 8.2.1. *For a function g defined on $\mathbb{R}^2$ and for a real number $\lambda > 0$ the **Peetre's maximal function** (see Lemma 2.1 in* [22]*) is*

$$g_\lambda^*(x) = \sup_{y \in \mathbb{R}^2} \frac{|g(x-y)|}{(1+|y|)^{2\lambda}}, \quad x \in \mathbb{R}^2.$$

Lemma 8.2.2. *Let $g \in \mathcal{S}'(\mathbb{R}^2)$ be such that supp $(\hat{g}) \subseteq \{\xi \in \hat{\mathbb{R}}^2 : |\xi| \leq R\}$ for some $R > 0$. Then, for any real $\lambda > 0$ there exists a $C_\lambda > 0$ such that, for $|\alpha| = 1$,*

$$(\partial^\alpha g)_\lambda^*(x) \leq C_\lambda g_\lambda^*(x), \quad x \in \mathbb{R}^2.$$

Proof. Since $g \in \mathcal{S}'$ has compact support in the Fourier domain, g is regular. More precisely, by the Paley-Wiener-Schwartz theorem g is slowly increasing (at most polinomialy) and infinitely differentiable (e.g., Theorem 7.3.1 in [20]). Let γ be a function in the Schwartz class such that $\hat{\gamma}(\xi) = 1$ if $|\xi| \leq R$. Then, $\hat{\gamma}(\xi)\hat{g}(\xi) = \hat{g}(\xi)$ for all $\xi \in \hat{\mathbb{R}}^2$. Hence, $\gamma * g = g$ and $\partial^\alpha g = \partial^\alpha \gamma * g$. Moreover,

$$\begin{aligned} |\partial^\alpha g(x-y)| &= \left| \int_{\mathbb{R}^2} \partial^\alpha \gamma(x-y-z) g(z) dz \right| = \left| \int_{\mathbb{R}^2} \partial^\alpha \gamma(w-y) g(x-w) dw \right| \\ &\leq \int_{\mathbb{R}^2} |\partial^\alpha \gamma(w-y)| \, (1+|w-y|)^{2\lambda} (1+|y|)^{2\lambda} \frac{|g(x-w)|}{(1+|w|)^{2\lambda}} dw, \end{aligned}$$

because of the triangular inequality. Therefore,

$$|\partial^\alpha g(x-y)| \leq g_\lambda^*(x)(1+|y|)^{2\lambda} \int_{\mathbb{R}^2} |\partial^\alpha \gamma(w-y)| \, (1+|w-y|)^{2\lambda} dw.$$

Since $\gamma \in \mathcal{S}$, the last integral equals a finite constant c_λ, independent of y, and we obtain

$$|\partial^\alpha g(x-y)| \leq c_\lambda g_\lambda^*(x)(1+|y|)^{2\lambda},$$

which shows the desired result.

■

We have a relation between the Hardy-Littlewood maximal function and g_λ^*.

Lemma 8.2.3. *Let $\lambda > 0$ and $g \in \mathcal{S}'$ be such that supp $(\hat{g}) \subseteq \{\xi \in \hat{\mathbb{R}}^2 : |\xi| \leq R\}$ for some $R > 0$. Then, there exists a constant $C_\lambda > 0$ such that*

$$g_\lambda^*(x) \leq C_\lambda \left(\mathcal{M}(|g|^{1/\lambda})(x) \right)^\lambda, \quad x \in \mathbb{R}^2.$$

Proof. Since g is band-limited, g is differentiable on $\mathbb{R}^2$ (by the Paley-Wiener-Schwartz theorem), so we can consider the pointwise values of g. Let $x, y \in \mathbb{R}^2$ and $0 < \delta < 1$. Choose $z \in \mathbb{R}^2$ such that $z \in |B_\delta(x-y)|$. We apply the mean value theorem to g and the endpoints $x-y$ and z to get

$$|g(x-y)| \leq |g(z)| + \delta \sup_{w: w \in B_\delta(x-y)} (|\nabla g(w)|).$$

Taking the $(1/\lambda)^{\text{th}}$ power and integrating with respect to the variable z over $B_\delta(x-y)$, we obtain

$$|g(x-y)|^{1/\lambda} \leq \frac{c_\lambda}{|B_\delta(x-y)|} \int_{B_\delta(x-y)} |g(z)|^{1/\lambda} dz$$

$$+c_\lambda \delta^{1/\lambda} \sup_{w: w\in B_\delta(x-y)} (|\nabla g(w)|)^{1/\lambda}. \tag{8.1}$$

Since $B_\delta(x-y) \subset B_{\delta+|y|}(x)$,

$$\int_{B_\delta(x-y)} |g(z)|^{1/\lambda}\, dz \le \int_{B_{\delta+|y|}(x)} |g(z)|^{1/\lambda}\, dz \le \left|B_{\delta+|y|}(x)\right| \mathcal{M}(|g|^{1/\lambda})(x),$$

and the sup term on the right hand side of (8.1) is bounded by

$$\begin{aligned} \sup_{w: w\in B_{\delta+|y|}(x)} (|\nabla g(w)|)^{1/\lambda} &= \sup_{t:|t|<\delta+|y|} (|\nabla g(x-t)|)^{1/\lambda} \\ &\le (1+\delta+|y|)^2 \left[(\nabla g)^*_\lambda(x)\right]^{1/\lambda}. \end{aligned}$$

Substituting these last two inequalities in (8.1) yields

$$\begin{aligned} |g(x-y)|^{1/\lambda} &\le c_\lambda \frac{\left|B_{\delta+|y|}(x)\right|}{|B_\delta(x-y)|} \mathcal{M}(|g|^{1/\lambda})(x) \\ &\quad + c_\lambda \delta^{1/\lambda} (1+\delta+|y|)^2 \left[(\nabla g)^*_\lambda(x)\right]^{1/\lambda}, \end{aligned}$$

and since $\left|B_{\delta+|y|}(x)\right| / |B_\delta(x-y)| = (\delta+|y|)^2/\delta^2$, we get

$$\begin{aligned} |g(x-y)|^{1/\lambda} &\le c_\lambda \frac{(\delta+|y|)^2}{\delta^2} \mathcal{M}(|g|^{1/\lambda})(x) \\ &\quad + c_\lambda \delta^{1/\lambda} (1+\delta+|y|)^2 \left[(\nabla g)^*_\lambda(x)\right]^{1/\lambda}. \end{aligned}$$

Taking the λ^{th} power yields

$$\frac{|g(x-y)|}{(1+|y|)^{2\lambda}} \le c'_\lambda \left\{ \frac{1}{\delta^{2\lambda}} [\mathcal{M}(|g|^{1/\lambda})(x)]^\lambda + \delta \left[(\nabla g)^*_\lambda(x)\right] \right\},$$

since $\delta < 1$ implies $(1+\delta+|y|) \le 2(1+|y|)$. Taking δ small enough so that $c'_\lambda C_\lambda \delta < 1/4$ (where C_λ is the constant in Lemma 8.2.2) we obtain

$$g^*_\lambda(x) \le c_\lambda \mathcal{M}(|g|^{1/\lambda})(x) + \frac{1}{2} g^*_\lambda(x).$$

Assume for the moment that $g \in \mathcal{S}$, hence $g^*_\lambda(x) < \infty$. So, we can subtract the second term in the right-hand side of the previous inequality from the left-hand side of the previous inequality and complete the proof for $g \in \mathcal{S}$. To remove the assumption $g \in \mathcal{S}$, we apply a standard regularization argument to a $g \in \mathcal{S}'$ as done in p. 22 of [24] or in Lemma A.4 of [13]. Let $\gamma \in \mathcal{S}$ satisfy supp $\hat{\gamma} \subset B(0,1)$, $\hat{\gamma}(\xi) \ge 0$ and $\gamma(0) = 1$. By Fourier inversion $|\gamma(x)| \le 1$ for all $x \in \mathbb{R}^2$. For $0 < \delta < 1$, let $g_\delta(x) = g(x)\gamma(\delta x)$. Then, supp $\hat{g}_\delta$ is also compact, $g_\delta \in \mathcal{S}$, $|g_\delta| \le |g|$ for all $x \in \mathbb{R}^2$ and $g_\delta \to g$ uniformly on compact sets as $\delta \to 0$. Applying the previous result to g_δ and letting $\delta \to 0$ we obtain the result for general $\mathcal{S}'$.

■

Lemmata 8.2.2 and 8.2.3 are Peetre's inequality for $f \in \mathcal{S}'$ whose proofs can be found in the references above and we reproduce them for completeness.

Proof of Lemma 4.2.3. Let $g(x) = (\psi_{A^{-j}B^{-\ell}} * f)(x)$. Since ψ is band-limited, so is g. On one hand, since $j \geq 0$ and $2^{j-1}|y| \leq |B^\ell A^j y|$,

$$\begin{aligned} g_\lambda^*(t) &= \sup_{y\in\mathbb{R}^2} \frac{|g(t-y)|}{(1+|y|)^{2\lambda}} \geq \sup_{y\in\mathbb{R}^2} \frac{|(\psi_{A^{-j}B^{-\ell}} * f)(t-y)|}{(1+2^j|y|)^{2\lambda}} \\ &= \sup_{y\in\mathbb{R}^2} \frac{|(\psi_{A^{-j}B^{-\ell}} * f)(t-y)|}{2^{2\lambda}(2^{-1}+2^{j-1}|y|)^{2\lambda}} \\ &\geq 2^{-2\lambda} \sup_{y\in\mathbb{R}^2} \frac{|(\psi_{A^{-j}B^{-\ell}} * f)(t-y)|}{(1+|B^\ell A^j y|)^{2\lambda}} = 2^{-2\lambda} \left|(\psi_{j,\ell,\lambda}^{**})(t)\right|. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathcal{M}(|g|^{1/\lambda})(t) &= \sup_{r>0} \frac{1}{|B_r(t)|} \int_{B_r(t)} |(\psi_{A^{-j}B^{-\ell}} * f)(y)|^{1/\lambda} dy \\ &= \mathcal{M}(|(\psi_{A^{-j}B^{-\ell}} * f|^{1/\lambda})(t). \end{aligned}$$

The result follows from Lemma 8.2.3 with $t = x$. ■

To prove Lemma 4.2.4 we need the next

Lemma 8.2.4. *Let $i \geq j \geq 0$ and $0 < a \leq r$. Also, let Q and P be identified with (i,m,n) and (j,ℓ,k), respectively. Then, for all $N > 3r/a$, any sequence $\{s_P\}_{P\in\mathcal{Q}^{j,\ell}}$ of complex numbers and any $x \in Q$,*

$$(s_{r,N}^*)_Q := \left(\sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j|x_Q-x_P|)^N}\right)^{1/r} \leq C_{a,r}\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}} |s_P|^a \chi_P\right)(x)\right]^{1/a}.$$

Moreover, when $i = j$,

$$\sum_{P\in\mathcal{Q}^{j,\ell}} \left[(s_{r,N}^*)_P \tilde{\chi}_P(x)\right]^q \leq C_{a,r}\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}} (|s_P|\,\tilde{\chi}_P)^a\right)(x)\right]^{q/a}.$$

Proof. Identify (i,m,n) and (j,ℓ,k) with Q and P, respectively. Then, $x_Q = A^{-i}B^{-m}n$ and $x_P = A^{-j}B^{-\ell}k$. Let $\mathcal{Q}^{j,\ell} := \{Q_{j,\ell,k} : k \in \mathbb{Z}^2\}$, then $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^2$. Write $d_P = |x_Q - x_P|$. Thus, we bound the sum in the definition of $(s_{r,N}^*)_Q$ as

$$\sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j|x_Q-x_P|)^N} \leq \left(\sum_{P\in\mathcal{Q}^{j,\ell}:d_P\leq 1} + \sum_{P\in\mathcal{Q}^{j,\ell}:d_P>1}\right) \frac{|s_P|^r}{(1+2^j d_P)^N}.$$

Choose λ such that $N > 3\lambda/2 > 3r/a$. Then, the inequality $(2^j d_P)^N > (2^{3j/2}d_P)^\lambda$ holds whenever $d_P > 2^{j(3\lambda/2-N)/(N-\lambda)}$. So, the previous inequality is bounded by

$$\sum_{P\in\mathcal{Q}^{j,\ell}:d_P\leq 1} \frac{|s_P|^r}{(1+2^j d_P)^N} + \sum_{P\in\mathcal{Q}^{j,\ell}:d_P>1} \frac{|s_P|^r}{(2^{3j/2}d_P)^\lambda}.$$

Defining

$$\begin{aligned} D_0 &= \{k \in \mathbb{Z}^2 : \left|A^{-i}B^{-m}n - A^{-j}B^{-\ell}k\right| \leq 1\} \\ &= \{P \in \mathcal{Q}^{j,\ell} : d_P = |x_Q - x_P| \leq 1\} \end{aligned}$$

and

$$\begin{aligned} D_\nu &= \{k \in \mathbb{Z}^2 : 2^{\nu-1} < 2^{3j/2}\left|A^{-i}B^{-m}n - A^{-j}B^{-\ell}k\right| \le 2^\nu\} \\ &= \{P \in \mathcal{Q}^{j,\ell} : 2^{\nu-1} < 2^{3j/2}d_P \le 2^\nu\}, \quad \nu = 1,2,3,\ldots, \end{aligned}$$

we have that

$$\begin{aligned} \sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j d_P)^N} &\le \sum_{P\in D_0} |s_P|^r + 2^\lambda \sum_{\nu=1}^\infty \sum_{P\in D_\nu} \frac{|s_P|^r}{2^{\nu\lambda}} \\ &\le 2^\lambda \sum_{\nu=0}^\infty \sum_{P\in D_\nu} \frac{|s_P|^r}{2^{\nu\lambda}} \le 2^\lambda \sum_{\nu=0}^\infty 2^{-\nu\lambda}\left(\sum_{P\in D_\nu} |s_P|^a\right)^{r/a}, \end{aligned}$$

because $1 + 2^j d_P \ge 1$ and $a \le r$. Now, when $x \in Q_{i,m,n} = Q$ and $P \in D_\nu$ then, by the definition of D_ν, $P = Q_{j,\ell,k} \subset B_{2^{\nu-3j/2+2}}(x)$ (this holds because for $j \ge 0$ the diameter of any $P = Q_{j,\ell,k}$ is less than $\sqrt{5}$ and the intervals in the definition of the D_ν's are dyadic with $\nu \ge 0$). Thus,

$$\sum_{P\in D_\nu} |s_P|^a = |P|^{-1} \int_{B_{2^{\nu-3j/2+2}}(x)} \sum_{P\in D_\nu} |s_P|^a \chi_P(y)dy.$$

Hence, writing $\left|\tilde{B}\right| = |B_{2^{\nu-3j/2+1}}(x)| = \pi 2^{2\nu-3j+4}$ we have that for $x \in Q_{i,m,n} = Q$,

$$\begin{aligned} \sum_{P\in\mathcal{Q}^{j,\ell}} \frac{|s_P|^r}{(1+2^j d_P)^N} &\le 2^\lambda \sum_{\nu=0}^\infty 2^{-\nu\lambda}\left(\frac{|P|^{-1}\left|\tilde{B}\right|}{\left|\tilde{B}\right|}\int_{\tilde{B}} \sum_{P\in D_\nu} |s_P|^a \chi_P(y)dy\right)^{r/a} \\ &\le 2^\lambda \sum_{\nu=0}^\infty 2^{-\nu\lambda}\pi^{r/a}2^{4r/a}2^{2\nu r/a}\left(\mathcal{M}\left(\sum_{P\in D_\nu} |s_P|^a \chi_P\right)(x)\right)^{r/a} \\ &\le C_{r,a,\lambda}\left(\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}} |s_P|^a \chi_P\right)(x)\right)^{r/a}, \end{aligned}$$

because $|P|^{-1} = 2^{3j}$ and $2r/a < \lambda$.

To prove the second inequality multiply both sides by $\tilde{\chi}_Q(x)$, rise to the power q and sum over $Q \in \mathcal{Q}^{j,\ell}$ to get

$$\begin{aligned} \sum_{Q\in\mathcal{Q}^{j,\ell}} \left[(s^*_{r,N})_Q\tilde{\chi}_Q(x)\right]^q &\le C\sum_{Q\in\mathcal{Q}^{j,\ell}}\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}} |s_P|^a \chi_P\right)(x)\right]^{q/a}\tilde{\chi}^q_Q(x) \\ &= C\sum_{Q\in\mathcal{Q}^{j,\ell}}\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}} (|s_P|\,\tilde{\chi}_P)^a\right)(x)\right]^{q/a}\chi_Q(x) \\ &= C\left[\mathcal{M}\left(\sum_{P\in\mathcal{Q}^{j,\ell}} (|s_P|\,\tilde{\chi}_P)^a\right)(x)\right]^{q/a}, \end{aligned}$$

since $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^2$.

■

Proof of Lemma 4.2.4. Let λ be such that $N > 3\lambda/2 > 3\max(1, r/q, r/p)$. If $r < \min(q,p)$, choose $a = r$. Otherwise, if $r \geq \min(q,p)$, choose a such that $r/(\lambda/2) < a < \min(r,q,p)$. It is always possible to choose such an a since $\lambda/2 > \max(1, r/q, r/p)$ implies $r/(\lambda/2) < \min(r,q,p)$. In both cases we have that

$$0 < a \leq r < \infty, \quad \lambda > 2r/a, \quad q/a > 1, \quad p/a > 1.$$

The previous argument is similar to that of [5]. Then, by Lemma 8.2.4 and Theorem 4.2.2

$$\begin{aligned}
\left\| \mathbf{s}^*_{r,N} \right\|_{\mathbf{f}^{\alpha,q}_p(AB)} &= \left\| \left(\sum_{P \in \mathcal{Q}_{AB}} (|P|^{-\alpha} (s^*_{r,N})_P \tilde{\chi}_P)^q \right)^{1/q} \right\|_{L^p} \\
&= \left\| \left(\sum_{j \geq 0} \sum_{\ell=-2^j}^{2^j} |Q_{j,\ell}|^{-\alpha q} \sum_{P \in \mathcal{Q}^{j,\ell}} (s^*_{r,N})^q_P \tilde{\chi}^q_P \right)^{1/q} \right\|_{L^p} \\
&\leq C \left\| \left(\sum_{j \geq 0} \sum_{\ell=-2^j}^{2^j} |Q_{j,\ell}|^{-\alpha q} \left[\mathcal{M} \left(\sum_{P \in \mathcal{Q}^{j,\ell}} (|s_P| \, \tilde{\chi}_P)^a \right) \right]^{q/a} \right)^{1/q} \right\|_{L^p} \\
&= C \left\| \left(\sum_{j \geq 0} \sum_{\ell=-2^j}^{2^j} \left[\mathcal{M} \left(\sum_{P \in \mathcal{Q}^{j,\ell}} (|P|^{-\alpha} |s_P| \, \tilde{\chi}_P)^a \right) \right]^{q/a} \right)^{1/q} \right\|_{L^p} \\
&= C \left\| \left(\sum_{j \geq 0} \sum_{\ell=-2^j}^{2^j} \left[\mathcal{M} \left(\sum_{P \in \mathcal{Q}^{j,\ell}} (|P|^{-\alpha} |s_P| \, \tilde{\chi}_P)^a \right) \right]^{q/a} \right)^{a/q} \right\|_{L^{p/a}}^{1/a} \\
&\leq C \left\| \left(\sum_{j \geq 0} \sum_{\ell=-2^j}^{2^j} \left(\sum_{P \in \mathcal{Q}^{j,\ell}} (|P|^{-\alpha} |s_P| \, \tilde{\chi}_P)^a \right)^{q/a} \right)^{a/q} \right\|_{L^{p/a}}^{1/a} \\
&= C \left\| \left(\sum_{j \geq 0} \sum_{\ell=-2^j}^{2^j} \sum_{P \in \mathcal{Q}^{j,\ell}} (|P|^{-\alpha} |s_P| \, \tilde{\chi}_P)^q \right)^{a/q} \right\|_{L^{p/a}}^{1/a} \\
&= C \left\| \left(\sum_{j \geq 0} \sum_{\ell=-2^j}^{2^j} \sum_{P \in \mathcal{Q}^{j,\ell}} (|P|^{-\alpha} |s_P| \, \tilde{\chi}_P)^q \right)^{1/q} \right\|_{L^p} \\
&= C \left\| \mathbf{s} \right\|_{f^{\alpha,q}_p(AB)},
\end{aligned}$$

because $\mathcal{Q}^{j,\ell}$ is a partition of $\mathbb{R}^2$.

The reverse inequality is trivial since $|s_Q| \leq (s^*_{r,\lambda})_Q$ always holds.

■

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